Certifying Combinatorial Solving Using Cutting Planes with Strengthening Rules

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"Proof Complexity and Beyond"
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Based on joint work with Jeremias Berg, Bart Bogaerts, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Yong Kiam Tan, and Dieter Vandersande

In a Galaxy Far, Far Away from Oberwolfach...

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
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  Prove that solver implementation adheres to formal specification
  Current techniques cannot scale to this level of complexity

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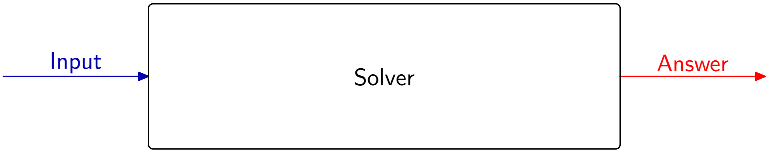
- **Proof logging**
  Make solver certifying [ABM$^+$11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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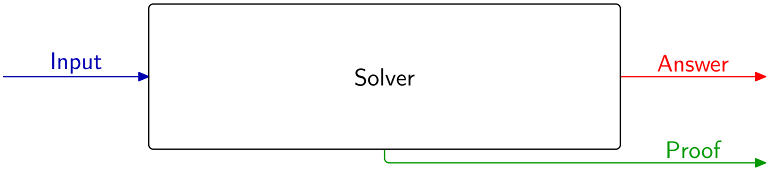

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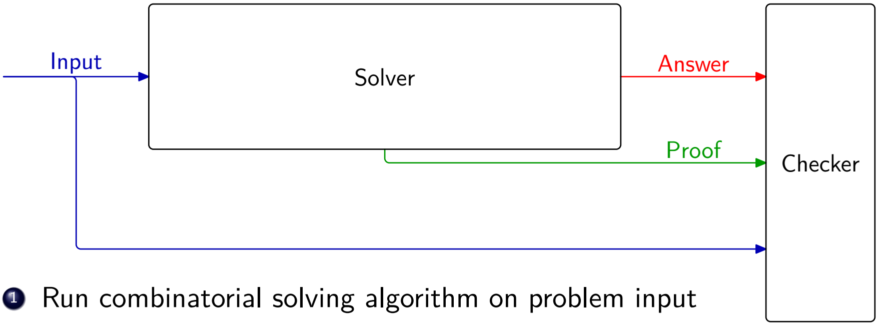

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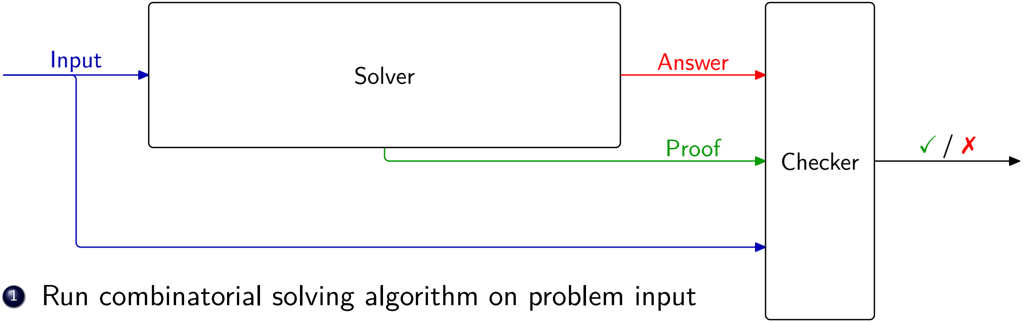
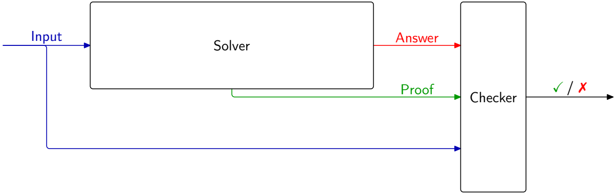


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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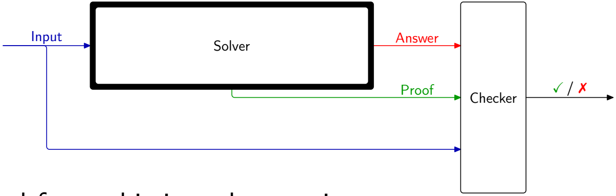

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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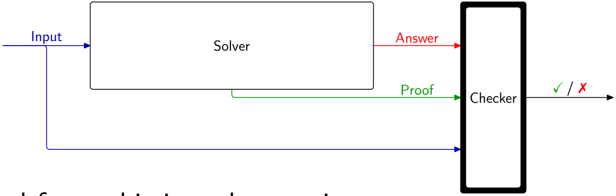

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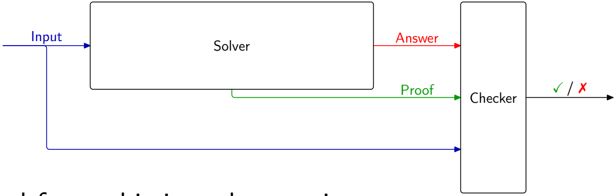

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Clear conflict expressivity vs. simplicity!

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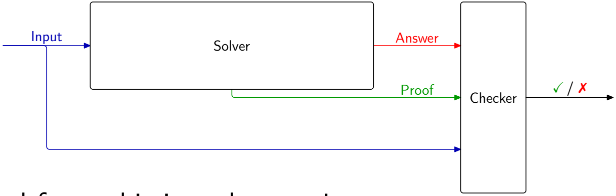

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Asking for both perhaps a little bit too good to be true?

This Talk

Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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1. Marketing pitch ☺

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1. Marketing pitch ☺
2. Highlight some interesting related questions in proof complexity

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [EG21, GMM+20, KM21, BBN+23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping) to solver code

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Performance goals

- Proof logging overhead small constant fraction ($\lessapprox 10\%$)
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

- just do proof logging [basically: add print statements to solver code]

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- do proof logging for 0-1 ILP formulation [but solver still works with original input]

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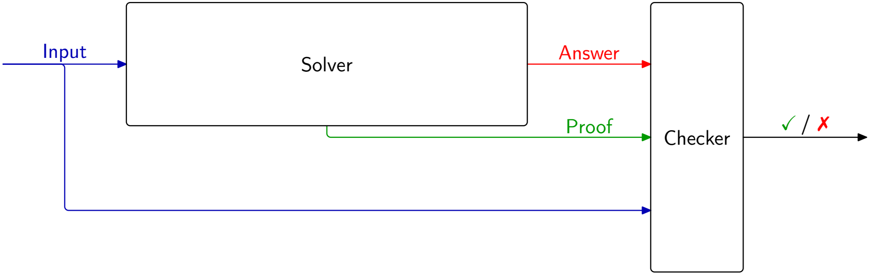
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

Proof Logging with Formally Verified Checking: Full Workflow



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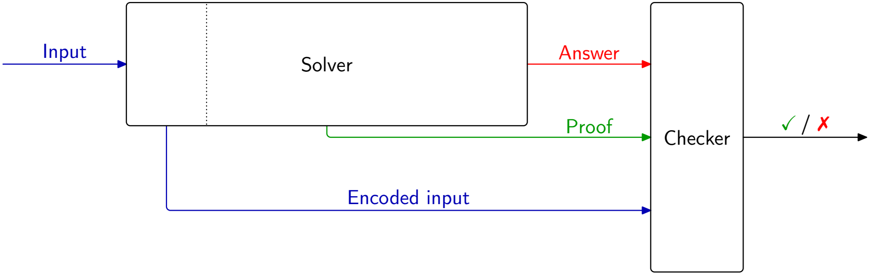

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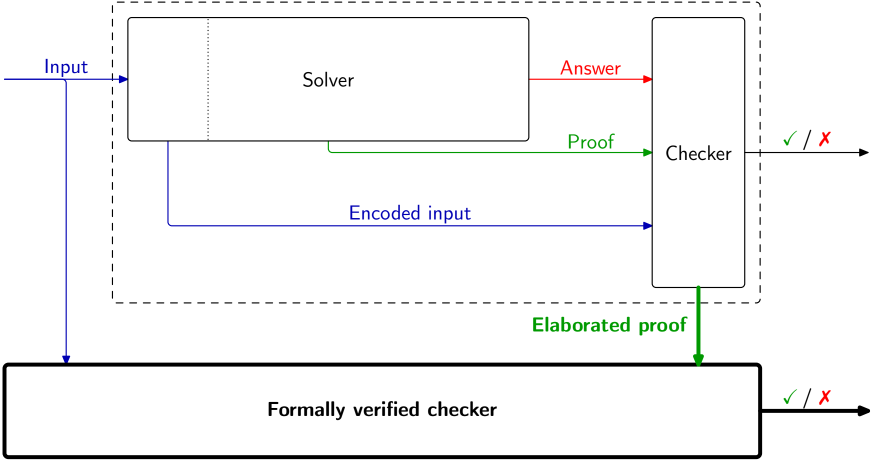

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
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Order $\mathcal{O}$

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set $\vec{z}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

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$$\dfrac{\text{Multiply by } 2 \quad \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add}$$

Cutting Planes Toy Example

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Multiply by 2
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Add
$$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Divide by 3
$$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\tfrac{1}{3}}$$

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$$\text{Divide by } 3\ \cfrac{\text{Add}\ \cfrac{\text{Add}\ \cfrac{\text{Multiply by } 2\ \cfrac{w+2x+y\geq 2}{2w+4x+2y\geq 4} \qquad w+2x+4y+2z\geq 5}{3w+6x+6y+2z\geq 9} \qquad \cfrac{\overline{z}\geq 0}{2\overline{z}\geq 0}\ \text{Multiply by } 2}{3w+6x+6y \qquad\qquad \geq 7}}{w+2x+2y\geq 3}$$

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By naming constraints by integers and literal axioms by the literal involved as

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$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Open Problem: Division Versus Saturation

Division $\dfrac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$

Saturation $\dfrac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$

How do division and saturation rules compare?

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Division $\dfrac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$ **Saturation** $\dfrac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$

How do division and saturation rules compare?

- Strengths of rules as such incomparable [GNY19]
- Cutting planes with division can be exponentially stronger than cutting planes with saturation
- Unknown whether cutting planes with saturation can be stronger than cutting planes with division

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \cup \{C\}$ are equisatisfiable

Want to allow adding such “redundant” constraints

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$

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C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_{\omega}$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

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   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(a + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Open Problems: Strength of Restricted Redundance Rules?

Adding redundance rule $\Rightarrow$ proof system polynomially equivalent to extended Frege

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Adding redundance rule $\Rightarrow$ proof system polynomially equivalent to extended Frege

1. What is the power of the redundance rule if we forbid new variables?
   For resolution + redundance known that:
   - Pigeonhole principle formulas easy
   - Tseitin formulas easy
2. What is the power of resolution with redundance if we only allow new variables $z \leftrightarrow C$ for previously derived clauses C?
   - Corresponds (kind of) to reasoning in core-guided MaxSAT solvers

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \models (\mathcal{C} \cup \mathcal{D} \cup \{C\})\restriction_\omega \cup \{f\restriction_\omega \leq f\}$$

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

Soundness of Dominance Rule

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Same inductive proof as before, but also nested forward induction over derivation
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Further extensions:

- Define dominance rule with respect to order $\mathcal{O}$ independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

Open Problem: Strength of Redundance and Dominance Rules?

Cutting planes with redundance and dominance is at least as strong as extended Frege

Could it be even stronger?!

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Could it be even stronger?!

Plausibly yes [KT23] — talk by Neil after the break

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Important to allow deletions of constraints from database

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- Satisfiable formulas can turn unsatisfiable(!)

Solution: distinguish between deletion from core set $\mathcal{C}$ and derived set $\mathcal{D}$

Deletion, Core Transfer, and Order Change

Deletion

1. Deletion of constraint C always OK from derived set $\mathcal{D}$
2. OK from core set $\mathcal{C}$ only if C can be rederived from $\mathcal{C} \setminus \{C\}$ with redundance rule (otherwise unchecked deletion — special conditions apply)

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Core transfer

Constraints from $\mathcal{D}$ can be moved to $\mathcal{C}$

Change of order

Possible to change order only if $\mathcal{D} = \emptyset$

Parity (XOR) Reasoning in SAT Solving

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
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Could add XORs to language, but prefer to keep things super-simple

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Use redundance rule with fresh variables a, b to derive

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(“$=$” syntactic sugar for “$\geq$” plus “$\leq$”)

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VeriPB can certify XOR reasoning [GN21]

Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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$$f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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2. Derive (for proof log only) pseudo-Boolean version of lex-leader constraint

$$f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

3. Derive symmetry breaking clauses from this PB constraint:

$$y_0 \qquad\qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

Open Problem: Symmetry Breaking with Redundance Rule?

Is the dominance rule really needed for fully general symmetry breaking?

Or could the redundance rule be enough?

Weaker DRAT strengthening rule sufficient for “pigeonhole-style” symmetries [HHW15]

Open Problem: Efficient Substitution Proofs?

Can cutting planes with redundance and dominance support proofs with lemmas/substitution efficiently?

Special case: symmetric learning in SAT solving [DBB17]

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Can cutting planes with redundance and dominance support proofs with lemmas/substitution efficiently?

Special case: symmetric learning in SAT solving [DBB17]

Seems very finicky. . .

Extension and substitution proof systems don't mix well

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

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Proof logging for other combinatorial problems and techniques

- Model counting
- Mixed integer linear programming *(work on SCIP in [CGS17, EG21, DEGH23])*
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And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23, GMM+24, HOGN24, IOT+24, MMN24]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Open:** Quite a few intriguing proof complexity questions (both upper and lower bounds)

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Thank you for your attention!

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