# Complexity Theory for Real-World Computation

Jakob Nordström

University of Copenhagen and Lund University

Complexity Days 2023
Paris, France
December 14, 2023

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# Three Simple Problems...

**Colouring**

Does the graph $G = (V, E)$ have a colouring with $k$ colours such that all neighbours have distinct colours?

# Three Simple Problems. . .

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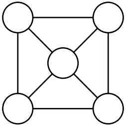

$3$-colouring?

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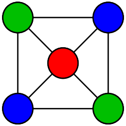

$3$-colouring? Yes

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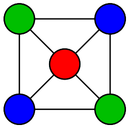

$3$-colouring? Yes, but no $2$-colouring

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**Clique**

Is there a clique in the graph $G = (V, E)$ with $k$ vertices that are all pairwise connected by edges in $E$?

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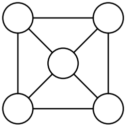

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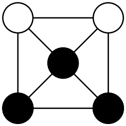

$3$-clique? Yes

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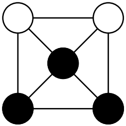

$3$-clique? Yes, but no $4$-clique

**Clique**

Is there a clique in the graph $G = (V, E)$ with $k$ vertices that are all pairwise connected by edges in $E$?

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Is there a clique in the graph $G = (V, E)$ with $k$ vertices that are all pairwise connected by edges in $E$?

### SAT

Given propositional logic formula, is there a satisfying assignment?

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$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means $x$ or $z$ should be true or $y$ false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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**SAT**

Given propositional logic formula, is there a satisfying assignment?

| | |
|---|---|
| COLOURING: | frequency allocation for mobile base stations |
| CLIQUE: | bioinformatics, computational chemistry |
| SAT: | easily models these and many other problems |

## . . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

# Solving NP in Theory and Practice

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- NP-complete, so probably very hard [Coo71, Lev73]
- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]

# Solving NP in Theory and Practice

- SAT mentioned in Gödel's letter in 1956 to von Neumann
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  - SAT [Hås01]
- Except that in practice, there are good algorithms for
  - COLOURING [DLMM08, DLMO09, DLMM11]
  - CLIQUE [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ$^+$01] that solve huge SAT formulas

How can we understand real-world algorithms for NP-hard problems?

**This talk:** Use proof complexity (not only conceivable answer)

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1. Is there a short proof using rules in this proof system?
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Study infeasible problems — proof of feasibility easy

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**Focus of this talk:** Question 1 for different proof systems/algorithms
Study infeasible problems — proof of feasibility easy

Question 2: Separate talk — lots of recent exciting progress; mostly negative (worst-case) results, e.g., [AM20, GKMP20, dRGN$^+$21]

# Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this talk**]
2. Get ideas for algorithmic improvements (e.g., [EN18, EN20, DGD$^{+}$21, DGN21, KBBN22])
3. Enhance algorithms to write machine-verifiable certificates of correctness (e.g., [EGMN20, GMN20, GMM$^{+}$20, GN21, GMN22, GMNO22, BGMN23, BBN$^{+}$23, GMM$^{+}$24]

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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake. . . ☺

# Outline

1. Conflict-Driven Clause Learning and Resolution
   - The SATISFIABILITY Problem in Different Shapes
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Approaches
   - Nullstellensatz
   - Gröbner Bases and Polynomial Calculus
   - Cutting Planes and Pseudo-Boolean Solving
3. Some Proof Systems We Won't Have Time for
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

# Formal Description of SAT Problem

- Variable $x$: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal $\ell$: variable $x$ or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

**The SATISFIABILITY (or just SAT) Problem**

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**The SATISFIABILITY (or just SAT) Problem**

Given a CNF formula $F$, is it satisfiable?

Here is our example formula again:

$$(x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
$$\wedge\ (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

# The Same Problem in Three Different Shapes

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$
\begin{gathered}
(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u) \\
\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)
\end{gathered}
$$

$$
\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}
$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

# The Same Problem in Three Different Shapes

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
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$$\begin{aligned} 1 - x - z + xz &= 0 \\ z - yz &= 0 \\ y - xy - yu + xyu &= 0 \\ yu &= 0 \\ 1 - u - v + uv &= 0 \\ xv &= 0 \\ u - uw &= 0 \\ xuw &= 0 \end{aligned}$$

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| | |
|---|---|
| $1 - x - z + xz = 0$ | $x + z \geq 1$ |
| $z - yz = 0$ | $y + (1 - z) \geq 1$ |
| $y - xy - yu + xyu = 0$ | $x + (1 - y) + u \geq 1$ |
| $yu = 0$ | $(1 - y) + (1 - u) \geq 1$ |
| $1 - u - v + uv = 0$ | $u + v \geq 1$ |
| $xv = 0$ | $(1 - x) + (1 - v) \geq 1$ |
| $u - uw = 0$ | $(1 - u) + w \geq 1$ |
| $xuw = 0$ | $(1 - x) + (1 - u) + (1 - w) \geq 1$ |

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$$\begin{aligned} x + z &\geq 1 \\ y - z &\geq 0 \\ x - y + u &\geq 0 \\ -y - u &\geq -1 \\ u + v &\geq 1 \\ -x - v &\geq -1 \\ -u + w &\geq 0 \\ -x - u - w &\geq -2 \end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0, 1\}$-valued solution?

# State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ$^+$01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

# Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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**Decision**

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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**Unit propagation**

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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Add to assignment trail

Until satisfying assignment or conflict

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

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**Decision**

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

**Unit propagation**

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

# Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

**Decision**

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# Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$
$u \overset{p \lor \overline{u}}{=} 0$
decision level 1

$\boxed{q \overset{\mathsf{d}}{=} 0}$
$r \overset{q \lor r}{=} 1$
$w \overset{\overline{r} \lor w}{=} 1$
decision level 2

$\boxed{x \overset{\mathsf{d}}{=} 0}$
$y \overset{u \lor x \lor y}{=} 1$
$z \overset{x \lor \overline{y} \lor z}{=} 1$
$\overset{\overline{y} \lor \overline{z}}{\bot}$
decision level 3

**Decision**

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

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Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

# Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$x \stackrel{\mathsf{d}}{=} 0$

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$$\left.\begin{array}{l} \boxed{p \overset{\mathsf{d}}{=} 0} \\ u \overset{p \lor \overline{u}}{=} 0 \end{array}\right\} \text{decision level 1}$$

$$\left.\begin{array}{l} \boxed{q \overset{\mathsf{d}}{=} 0} \\ r \overset{q \lor r}{=} 1 \\ w \overset{\overline{r} \lor w}{=} 1 \end{array}\right\} \text{decision level 2}$$

$$\left.\begin{array}{l} \boxed{x \overset{\mathsf{d}}{=} 0} \\ y \overset{u \lor x \lor y}{=} 1 \\ z \overset{x \lor \overline{y} \lor z}{=} 1 \\ \overset{\overline{y} \lor \overline{z}}{\bot} \end{array}\right\} \text{decision level 3}$$

Could backtrack by removing last decision level & flipping last decision

# Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

| | |
|---|---|
| $p \stackrel{\mathsf{d}}{=} 0$ | decision level 1 |
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| $q \stackrel{\mathsf{d}}{=} 0$ | decision level 2 |
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| $w \stackrel{\overline{r} \lor w}{=} 1$ | |
| $x \stackrel{\mathsf{d}}{=} 0$ | decision level 3 |
| $y \stackrel{u \lor x \lor y}{=} 1$ | |
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| $\stackrel{\overline{y} \lor \overline{z}}{\bot}$ | |

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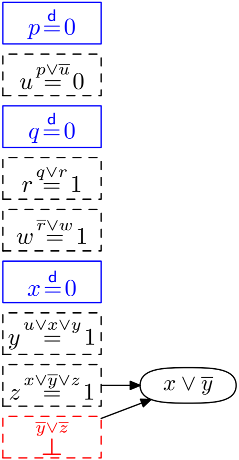
But want to learn from conflict and cut away as much of search space as possible

# Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$




Could backtrack by removing last decision level & flipping last decision

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Case analysis over $z$ for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
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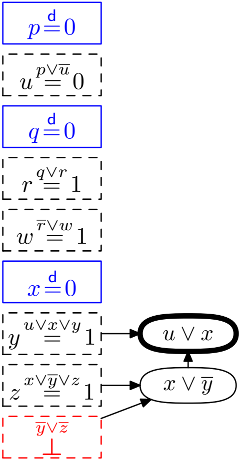
- Resolve clauses by merging them & removing $z$ — must satisfy $x \vee \overline{y}$

# Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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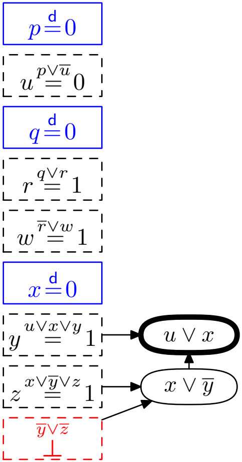
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing $z$ — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable after last decision — learn and backjump

# Complete Toy Example for CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

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$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$ → $u \lor x$

$z \overset{x \lor \overline{y} \lor z}{=} 1$ → $x \lor \overline{y}$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

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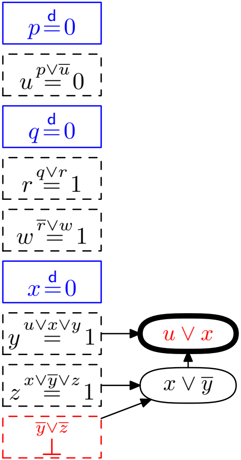
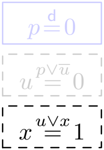
Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

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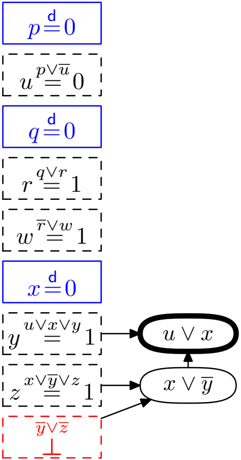
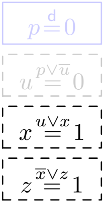
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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Then continue as before...

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$p \stackrel{\mathsf{d}}{=} 0$

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$u \lor x$

$x \lor \overline{y}$

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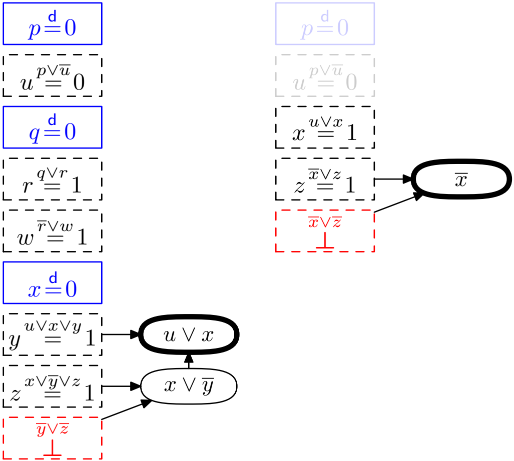

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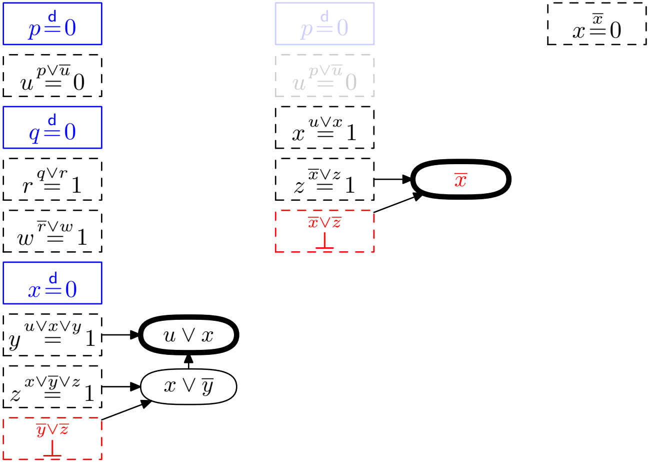

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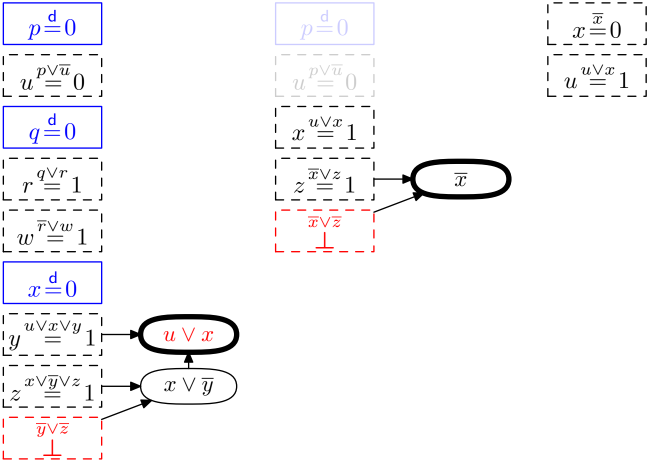

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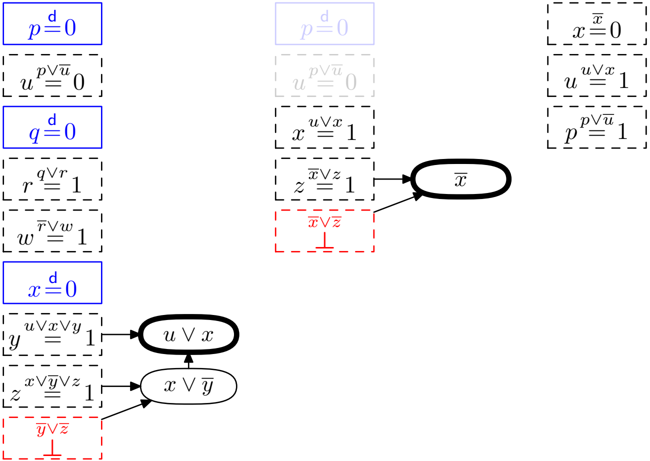

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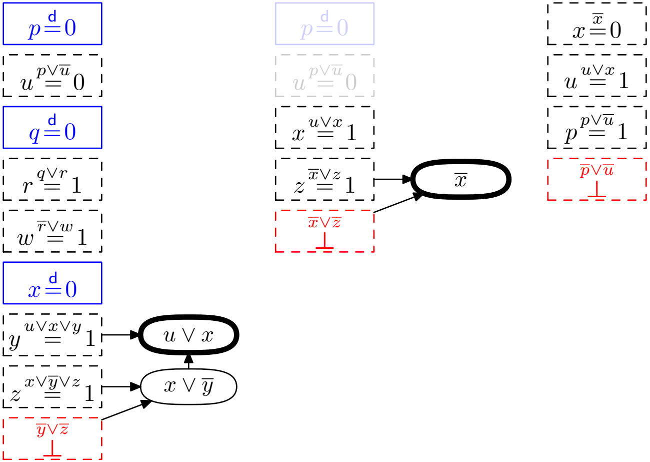

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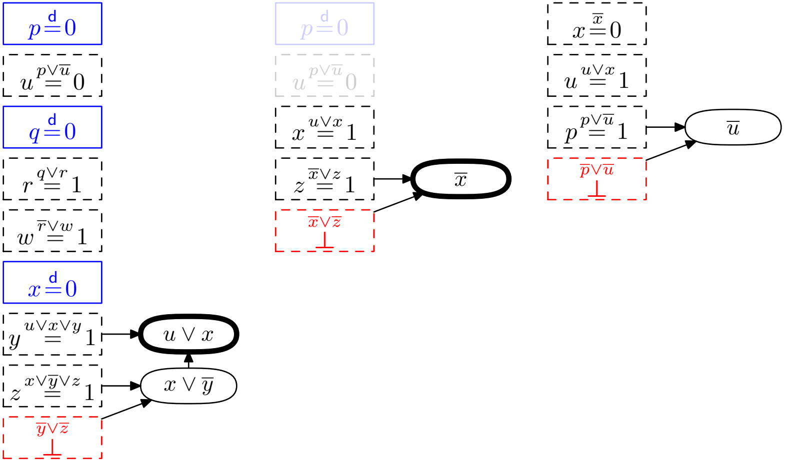

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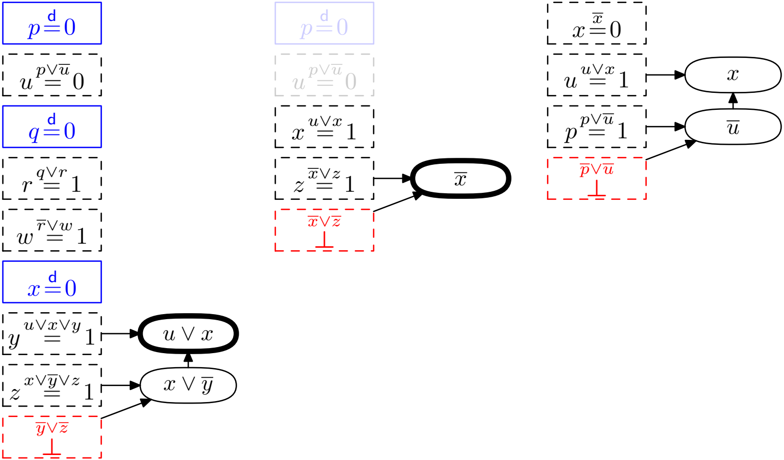

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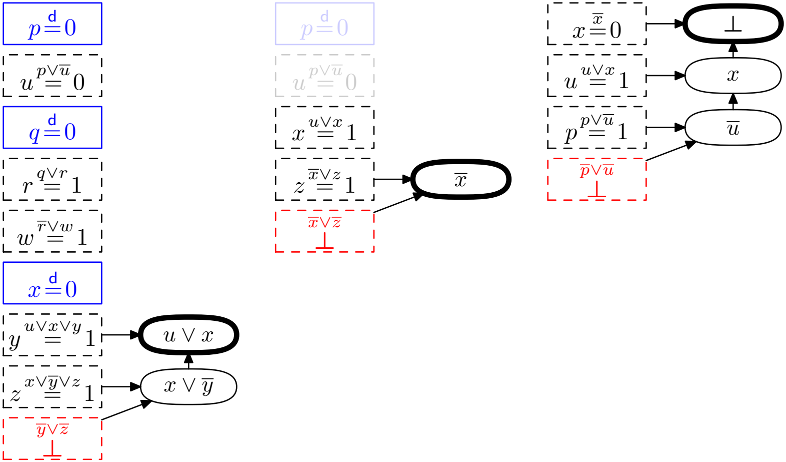

# SAT Solver Analysis and the Resolution Proof System

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**Resolution proof system [Bla37, Rob65]**

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

# Resolution Proofs by Contradction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

**Observation**

*If $F$ is a satisfiable CNF formula and $D$ is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.*

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*If $F$ is a satisfiable CNF formula and $D$ is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.*

So can prove $F$ unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from $F$ by resolution

Such proof by contradiction also called resolution refutation

# CDCL and Resolution Proofs

Obtain resolution proof...

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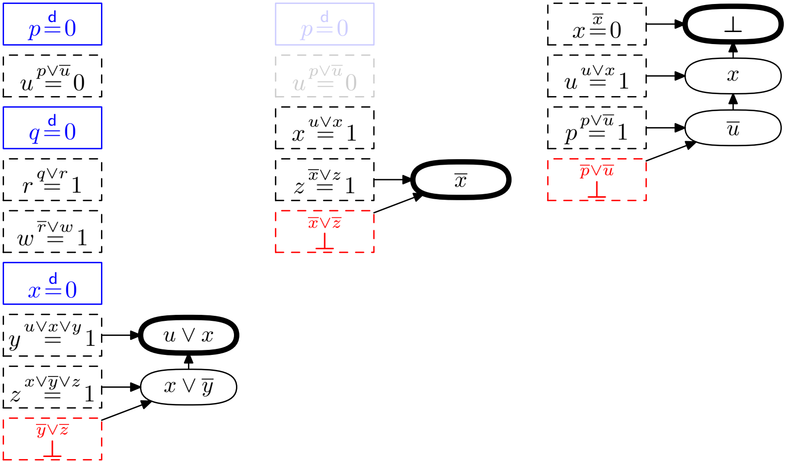
Obtain resolution proof from our example CDCL execution...



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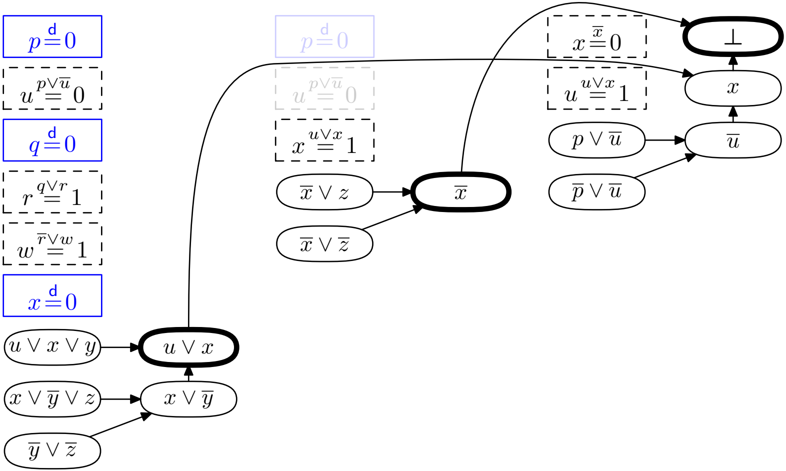
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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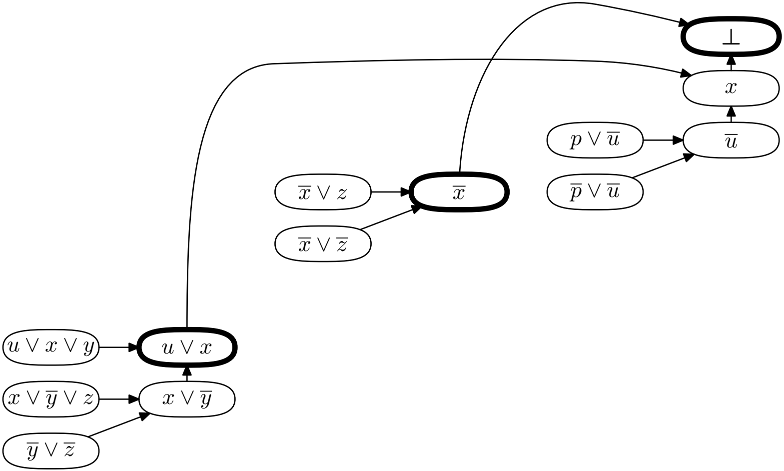

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- Hence, lower bounds on resolution proof size ⇒ lower bounds on CDCL running time
- Lower (and upper) bounds for different methods of reasoning about propositional logic formulas studied in proof complexity

(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

# Examples of Hard Formulas For Resolution (1/3)

**Pigeonhole principle (PHP) formulas** [Hak85]

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**Pigeonhole principle (PHP) formulas** [Hak85]

"$n+1$ pigeons don't fit into $n$ holes"

Variables $p_{i,j}$ = "pigeon $i \to$ hole $j$"; $1 \leq i \leq n+1$; $1 \leq j \leq n$

| | |
|---|---|
| $p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$ | every pigeon $i$ gets a hole |
| $\overline{p}_{i,j} \lor \overline{p}_{i',j}$ | no hole $j$ gets two pigeons $i \neq i'$ |

Can also add "functionality" and "onto" axioms

| | |
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| $p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$ | every hole $j$ gets a pigeon |

# Examples of Hard Formulas For Resolution (1/3)

**Pigeonhole principle (PHP) formulas** [Hak85]
"$n+1$ pigeons don't fit into $n$ holes"

Variables $p_{i,j}$ = "pigeon $i \to$ hole $j$"; $1 \leq i \leq n+1$; $1 \leq j \leq n$

| | |
|---|---|
| $p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$ | every pigeon $i$ gets a hole |
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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\bigl(\Omega\bigl(\sqrt[3]{N}\bigr)\bigr)$ clauses
(measured in terms of formula size $N$)

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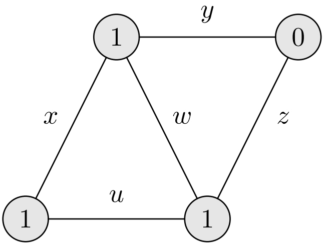
Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



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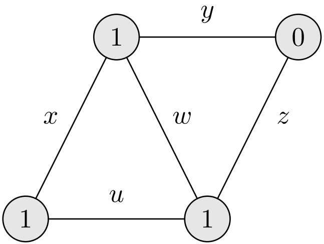


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\begin{array}{llll}
 & (u \lor x) & \land & (y \lor \overline{z}) \\
\land & (\overline{u} \lor \overline{x}) & \land & (\overline{y} \lor z) \\
\land & (w \lor x \lor y) & \land & (u \lor w \lor z) \\
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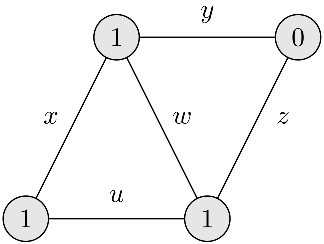


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$$

Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\bmod 2$**"**

# Examples of Hard Formulas for Resolution (3/3)

**Random $k$-CNF formulas** [CS88]

$\Delta n$ randomly sampled $k$-clauses over $n$ variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable $3$-CNF almost surely)

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**And more...**

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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**But not CLIQUE!**

- Refuting existence of $k$-clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR⁺21]

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- Add Boolean axioms

$$x_j^2 - x_j = 0$$

for all variables

# Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$
\begin{aligned}
p_1(x_1, \ldots, x_n) &= 0 & \qquad x_1^2 - x_1 &= 0 \\
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&\vdots & &\vdots \\
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in polynomial ring over field $\mathbb{F}$

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## Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1,\ldots,x_n) \cdot p_i(x_1,\ldots,x_n) + \sum_{j=1}^{n} r_j(x_1,\ldots,x_n) \cdot (x_j^2 - x_j) = 1$$

# Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

# Nullstellensatz Example (Not Expanded out)

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
$$xuw$$

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& (1-y) \cdot (1-x)(1-z) \\
+\ & (1-x) \cdot (1-y)z \\
+\ & 1 \cdot (1-x)y(1-u) \\
+\ & (1-x) \cdot yu \\
+\ & x \cdot (1-u)(1-v) \\
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\end{aligned}$$

Size $27$
Degree $3$
(No use of Boolean axioms)

# Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials $q_i$, $r_j$ as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

# Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1-x_1)(1-x_2)(1-x_3) = 1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$$

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$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

# Dynamic Construction of Nullstellensatz Certificates

## Nullstellensatz again

Infeasibility of

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot q \in \mathcal{I}$ for any $r$
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

# Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials $m, m', t$:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

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"Multivariate division": Reduce $p$ modulo all $q$ in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

# Gröbner Bases: Buchberger's Algorithm

**Buchberger's algorithm for computing Gröbner bases (very rough)**

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
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**Facts:**

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

# Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by $p_i$, $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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- A refutation is a derivation ending with the polynomial $1$
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation

$$\cfrac{x'yz' \qquad \cfrac{\cfrac{yz}{x'yz} \qquad \cfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

# Polynomial Calculus is Strictly Stronger than Resolution

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- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\bmod\ p$ if $p \neq$ field characteristic [BGIP01]
- Random $k$-CNF formulas
  - all characteristics except $2$ [BI99]
  - all characteristics [AR03]

# Colouring and Clique for Polynomial Calculus

### Colouring

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN$^+$23]

### Clique

Essentially nothing known!

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- Use dual variables! [KBBN22]

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2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

# Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

# SAT as System of $0$–$1$ Integer Linear Inequalities

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

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- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

for all variables

# Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of $0$-$1$ integer linear inequalities

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## Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

# Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of $0$-$1$ integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A, \sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil$
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

# Cutting Planes vs. Resolution

- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger
  (e.g., for PHP, just count and argue that #pigeons $>$ #holes)

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger
  (e.g., for PHP, just count and argue that #pigeons > #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
\land&(x_1 \lor x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor x_4 \lor x_6) \land (x_1 \lor x_2 \lor x_5 \lor x_6) \\
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# Hard Formulas for Cutting Planes

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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of $m$-clique; $1 \leq k \leq m$, $1 \leq i \leq n$
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| | |
|---|---|
| $q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n}$ | some vertex is the $k$th member of clique |
| $\overline{q}_{k,i} \vee \overline{q}_{k',i}$ | clique members are uniquely defined ($k \neq k'$) |
| $p_{i,j} \vee \overline{q}_{k,i} \vee \overline{q}_{k',j}$ | clique members are connected by edges |
| $r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1}$ | every vertex $i$ has a colour |
| $\overline{p}_{i,j} \vee \overline{r}_{i,\ell} \vee \overline{r}_{j,\ell}$ | neighbours have distinct colours |

# More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
- Hence, no small cutting planes proofs can exist either

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Some exciting contributions in [HP17, FPPR22, GGKS20, Sok23]

Nothing known for COLOURING or CLIQUE
Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

# SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

Perhaps counter-intuitively, hard to make competitive with CDCL

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**Challenge 1: Conjunctive normal form**

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- Solvers can rewrite CNF to more helpful $0$-$1$ linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

# Division Versus Saturation

Use negated literals as needed to get all $a_i$, $A$ positive

## Boolean derivation rules for $0$–$1$ integer linear inequalities

$$\text{Division}\ \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A / c \rceil} \quad c \in \mathbb{N}^+$$

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- ... And most often also in practice [EN18]

# Sherali-Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

**Nullstellensatz**

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# SA, SoS, and Other Proof Systems

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Strict hierarchy (over $\mathbb{R}$):

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Sums of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$) while Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system, except it cannot do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] is recommended for more reading

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Stabbing planes with polynomial-size coefficient can be simulated by cutting planes with quasi-polynomial overhead [DT20, FGI+21]

# Extended Resolution [Tse68]

**Resolution rule**

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

**Extension rule** introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable $a$ (encoding that $a \leftrightarrow (x \wedge y)$ must hold)

# Extended Resolution and SAT Solving

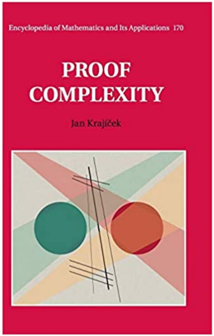
- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

# Some More References for Further Reading

**Handbook of Satisfiability**
(Especially chapter 7 ☺)


[BHvMW21]

**Proof Complexity**
by Jan Krajíček




[Kra19]

# Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ Conflict-driven clause learning
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Thank you for your attention!

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