How Limited Interaction Hinders Real Communication
(and What It Means for Proof and Circuit Complexity)

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Property Testing and Combinatorics
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Joint work with Susanna F. de Rezende and Marc Vinyals

SAT in Theory. . .

The Satisfiability Problem (SAT)

Given a formula F in conjunctive normal form (CNF), can the variables be assigned so as to satisfy all constraints?

- Has played leading role in TCS ever since discovery of NP-completeness in [Coo71, Lev73]
- Convential wisdom: this is a **very** hard problem indeed (Exponential Time Hypothesis [IP01] standard assumption)
- Yet essentially no nontrivial time complexity lower bounds
- More limited goal of time-space trade-offs also not very successful: E.g. SAT cannot be decided in time $n^{1.8}$ and space $n^{o(1)}$ [Wil08]
- Not only a sign of our weakness — there is a formidable adversary. . .

. . . and in Practice

- Enormous progress on applied SAT algorithms last 15-20 years
- Current state-of-the-art SAT solvers can deal with real-world instances containing millions of variables
- Use methods such as
  - conflict-driven clause learning (CDCL)
  - Gaussian elimination
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- **This work:** First such strong trade-offs capturing also cutting planes

Informal Statement of Results

Theorem (Main)

First time-space trade-offs holding uniformly for resolution, polynomial calculus, and cutting planes for formulas such that:

- $\exists$ *proofs in small size*
- $\exists$ *proofs in small total space*
- $\forall$ *proofs few formulas in memory* $\Rightarrow$ *length exponential*

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But we need **communication complexity** to attack cutting planes

Outline

1. Proof Complexity
   - Preliminaries
   - Previous Work
   - Our Results
2. Tools and Techniques
   - Communication Complexity
   - Pebbling Formulas
   - Lifting/Composition of Search Problems
   - Dymond–Tompa Game
3. Open Problems

Some Terminology and Notation

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
- k-CNF formula: all clauses of size $\leq k = \mathcal{O}(1)$
- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)

The Theoretical Model

- Proof system operates with formulas of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
- Refutation ends when (explicit) contradiction is derived

Cutting Planes (CP)

Clauses interpreted as linear inequalities
E.g., $x \vee y \vee \overline{z} \rightsquigarrow x + y + (1 - z) \geq 1 \rightsquigarrow x + y - z \geq 0$

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Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula/system of inequalities unsatisfiable

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} \lor x_{1,2}$
2. $x_{2,1} \lor x_{2,2}$
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Pigeonhole principle (PHP)
"$n+1$ pigeons don't fit into n holes"

Variables $x_{i,j}$ = "pigeon i goes into hole j"

$x_{i,1} \lor x_{i,2} \lor \cdots \lor x_{i,n}$	every pigeon i gets a hole
$\overline{x}_{i,j} \lor \overline{x}_{i',j}$	no hole j gets two pigeons $i \neq i'$

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5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 4: $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$

$$-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$$
$$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 4: $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$

$$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 4: $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 4: $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$$
$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$

$-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 5: $-x_{1,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{3,1} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-x_{1,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-x_{1,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,1} - x_{2,1} \geq -1$
Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-x_{1,2} - x_{3,2} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Erase the line $-x_{1,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} \geq -1$$
$$-x_{1,2} - x_{3,2} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 6: $-x_{2,1} - x_{3,1} \geq -1$
Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$

Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - x_{2,1} - x_{3,1} \geq -2$
Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$-x_{2,2} - x_{3,2} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$-x_{2,2} - x_{3,2} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
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Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Erase the line $-x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
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7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
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Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

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2. $x_{2,1} + x_{2,2} \geq 1$
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4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,1} - 2x_{2,1} - 2x_{3,1} \geq -3$
Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
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Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 7: $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 8: $-x_{1,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
$-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$$
$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{1,2} - x_{2,2} \geq -1$
Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Write down axiom 9: $-x_{2,2} - x_{3,2} \geq -1$
Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$

Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$

Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$

Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Erase the line $x_{2,1} + x_{2,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{2,1} + x_{2,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$x_{1,1} + x_{1,2} \geq 1$$
$$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} \geq 1$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Divide to get $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$

$$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$$
$$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Erase the line $-2x_{1,2} - 2x_{2,2} - 2x_{3,2} \geq -3$
Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
Erase the line $-x_{1,2} - x_{2,2} - x_{3,2} \geq -1$
Erase the line $-x_{1,1} - x_{2,1} - x_{3,1} \geq -1$
Write down axiom 1: $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 2: $x_{2,1} + x_{2,2} \geq 1$
Add to get $x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
Erase the line $x_{2,1} + x_{2,2} \geq 1$
Erase the line $x_{1,1} + x_{1,2} \geq 1$
Write down axiom 3: $x_{3,1} + x_{3,2} \geq 1$

$-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
$x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} \geq 2$
$x_{3,1} + x_{3,2} \geq 1$

Example: CP Refutation of Pigeonhole Principle

1. $x_{1,1} + x_{1,2} \geq 1$
2. $x_{2,1} + x_{2,2} \geq 1$
3. $x_{3,1} + x_{3,2} \geq 1$
4. $-x_{1,1} - x_{2,1} \geq -1$
5. $-x_{1,1} - x_{3,1} \geq -1$
6. $-x_{2,1} - x_{3,1} \geq -1$
7. $-x_{1,2} - x_{2,2} \geq -1$
8. $-x_{1,2} - x_{3,2} \geq -1$
9. $-x_{2,2} - x_{3,2} \geq -1$

History of derivation steps

Add to get $-x_{1,1} - x_{2,1} - x_{3,1} - x_{1,2} - x_{2,2} - x_{3,2} \geq -2$
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Complexity Measures for Cutting Planes

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Line space = max # lines in memory during refutation

Total space = max # bits in memory (sum also size of coefficients)

Hardness Results for Cutting Planes

Clique-coclique formulas

"A graph with an m-clique is not $(m-1)$-colourable"

Exponential lower bound via interpolation and circuit complexity [Pud97]

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Tseitin formulas

"Sum of degrees of vertices in graph is even"

Short refutations of (lifted) Tseitin formulas on expanders must have large space [GP14]

Long-standing open problems to show such refutations don't exist

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- Short refutations of some so-called pebbling formulas must have large space [HN12, GP14] (and such refutations do exist)

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What about "true" trade-offs?

Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)

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Theorem (Informal sample)

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5. Construct graphs G with strong round-cost trade-offs for Dymond–Tompa pebbling

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- Function f solved by r-round real communication in cost c if $\exists$ protocol such that
  - $\#$ rounds $\leq r$
  - total $\#$ comparisons made by referee $\leq c$

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  - Bob with private input y
  - Both deterministic but have unbounded computational powers
- Task: compute $f(x, y)$ by sending messages to referee
- Method: In each round v
  - Alice sends $a_{v,1}(x), \ldots, a_{v,c_v}(x) \in \mathbb{R}^{c_v}$
  - Bob sends $b_{v,1}(y), \ldots, b_{v,c_v}(y) \in \mathbb{R}^{c_v}$
  - Referee announces results of comparisons $a_{v,i}(x) \leq b_{v,i}(y)$ for $i \in [c_v]$
- Function f solved by r-round real communication in cost c if $\exists$ protocol such that
  - $\#$ rounds $\leq r$
  - total $\#$ comparisons made by referee $\leq c$
- Strictly stronger than standard deterministic communication

Falsified Clause Search Problem

Fix:

- unsatisfiable CNF formula F
- (devious) partition of $Vars(F)$ between Alice and Bob

Falsified clause search problem $Search(F)$

Input: Assignment α to $Vars(F)$ split between Alice and Bob

Output: Clause $C \in F$ such that α falsifies C

Actually, computing not function but relation — will mostly ignore this for simplicity

Succinct Refutations Yield Efficient Protocols

Evaluate blackboard configurations of a refutation of F under α

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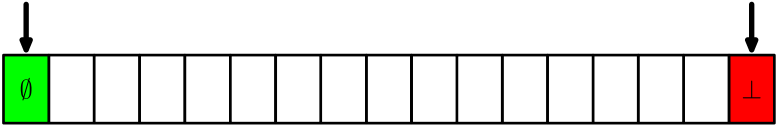

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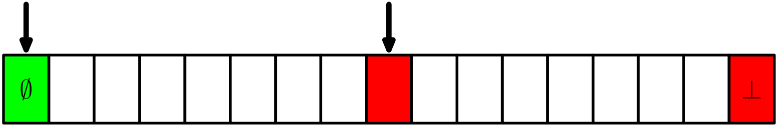

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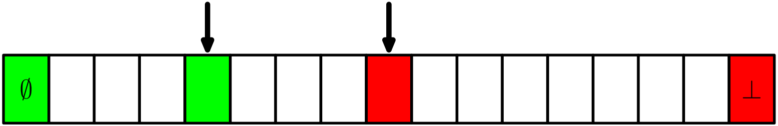

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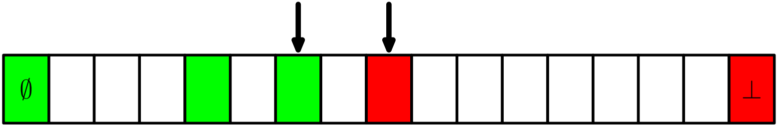

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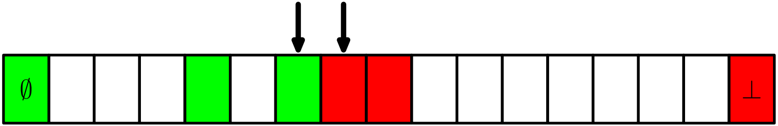

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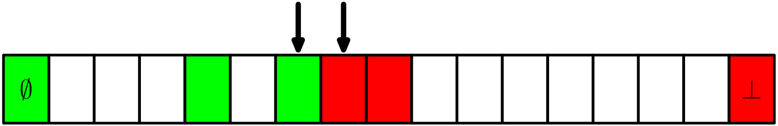

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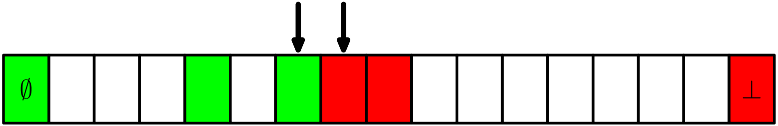

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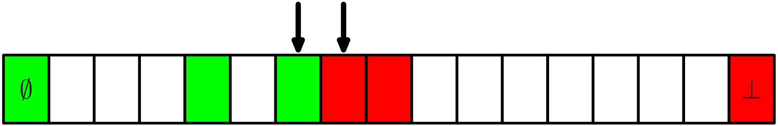

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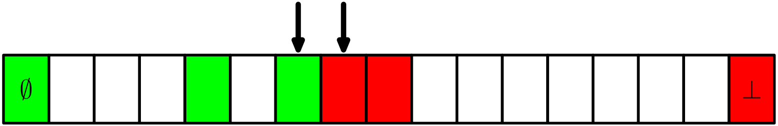

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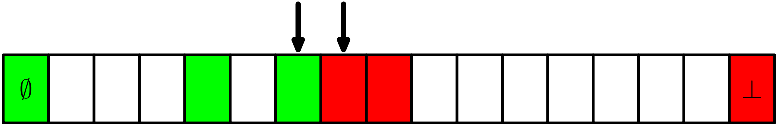

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(Alice and Bob simply evaluate their parts of each inequality and ask referee to compare)

Where to Get Formulas with Trade-off Properties?

Questions about time-space trade-offs fundamental in theoretical computer science

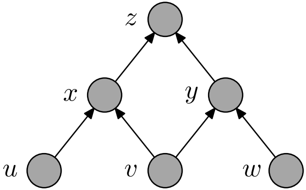
Well-studied (and well-understood) for pebble games modelling calculations described by DAGs

In particular, for black-white pebble game investigated by [CS76] and many others

Pebbling Contradiction

CNF formulas encoding black-white pebble game played on DAG G

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



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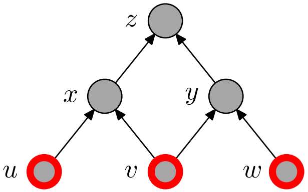

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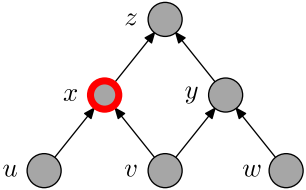

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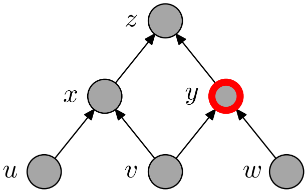

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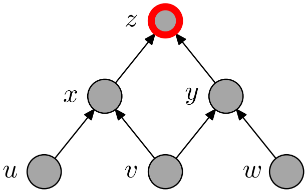

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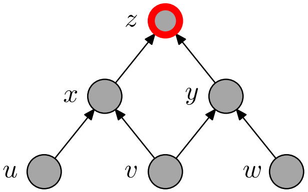

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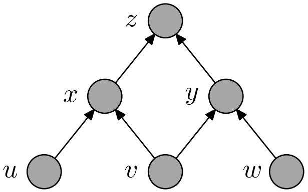

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Appeared in various contexts in e.g. [RM99, BEGJ00, BW01]

Used in [Nor06, NH08, BN08, BN11, BNT13] to study space and size-space trade-offs in resolution and polynomial calculus

Inherit some DAG properties, but not enough — make formulas harder!

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Construct hard communication problems by "hardness amplification" using lifting or composition

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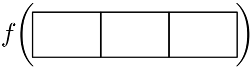

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Start with function $f : \{0,1\}^m \to \{0,1\}$

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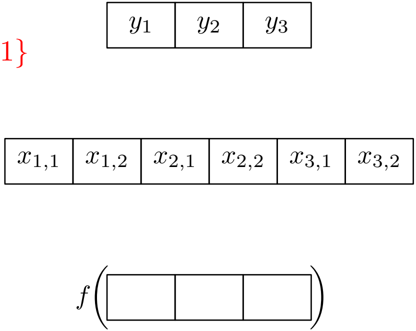

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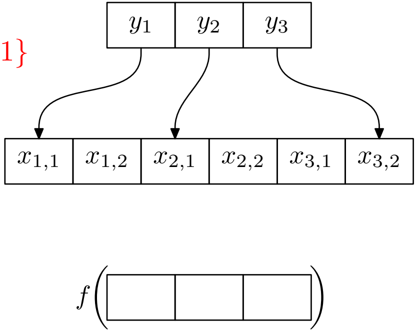

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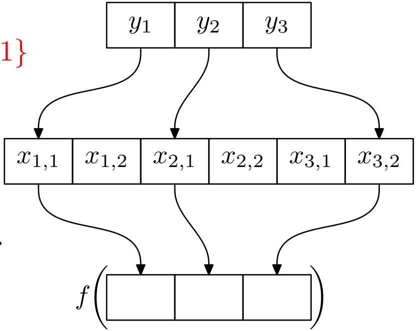

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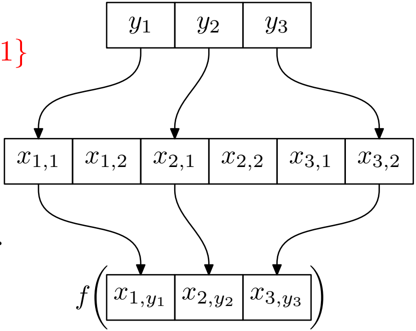
Length-ℓ lifting of f defined as $Lift_\ell(f)(x,y) := f(x_{1,y_1}, \ldots, x_{m,y_m})$



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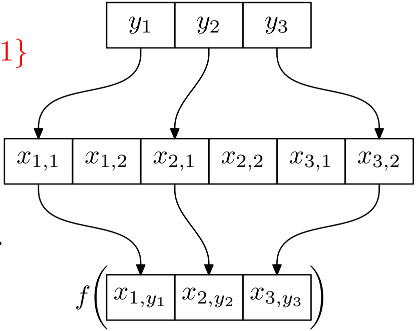

Building on ideas from e.g. [She08, BHP10]

Simulation of Protocols by Parallel Decision Trees [Val75]

Each node t in tree labelled by variables V_t;
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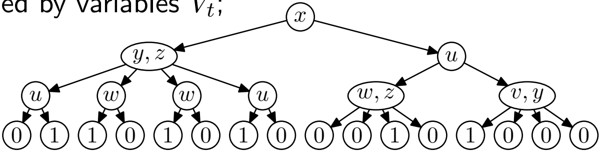

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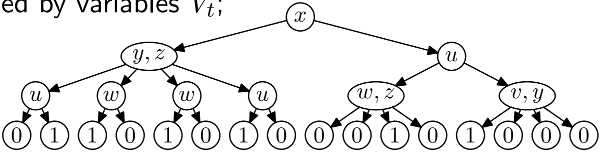

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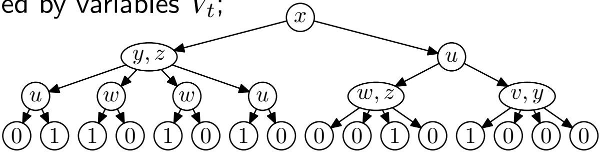

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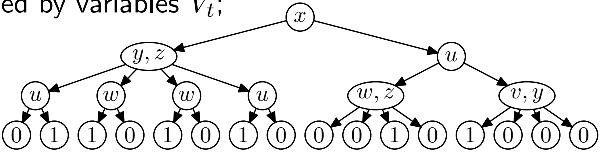

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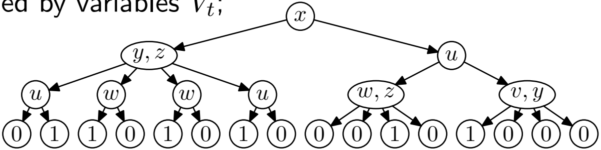

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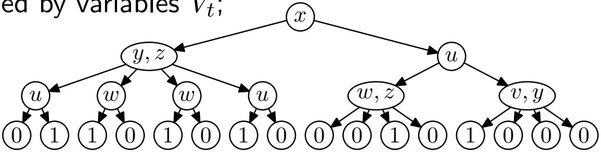

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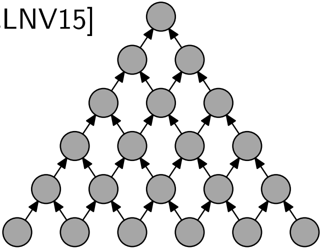
Simulation theorem of protocol by decision tree (hard direction)

Let S search problem with domain $\{0,1\}^m$ and let $\ell = m^{3+\epsilon}$, $\epsilon > 0$. Then:
$\exists$ r-round real communication protocol in cost c solving $Lift_\ell(S)$
$\Rightarrow \exists$ depth-r parallel decision tree solving S width $\mathcal{O}(c/\log \ell)$ queries.

From Parallel Decision Trees to Dymond–Tompa Games

- From [DT85]; recently studied in [Cha13, CLNV15]



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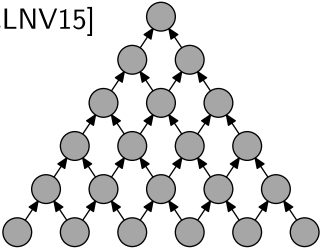

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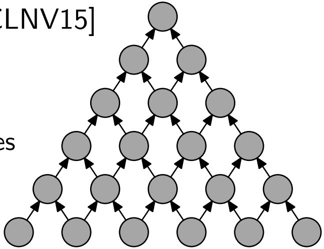

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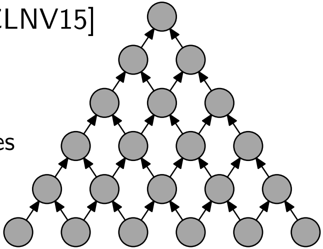

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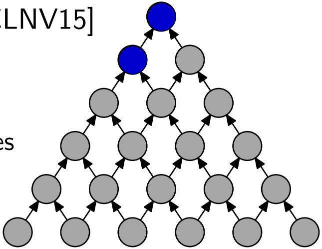

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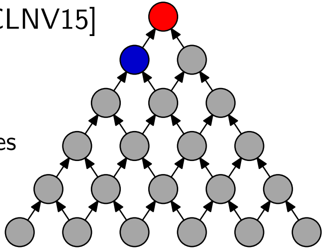

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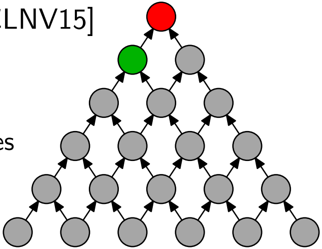

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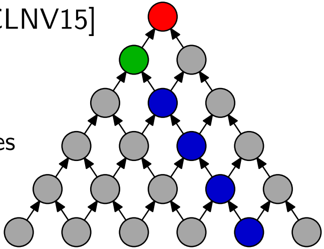

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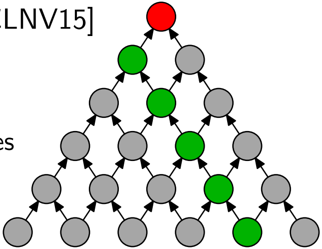

From Parallel Decision Trees to Dymond–Tompa Games

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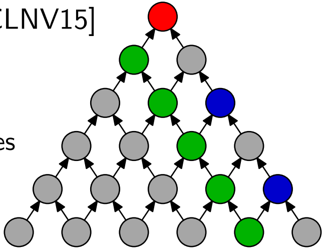

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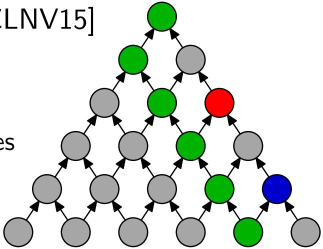

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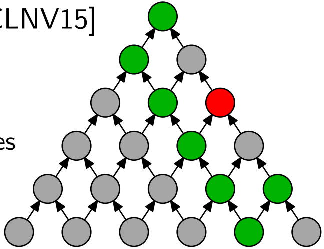

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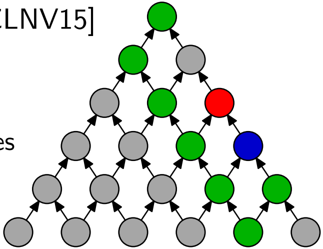

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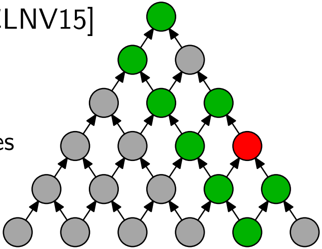

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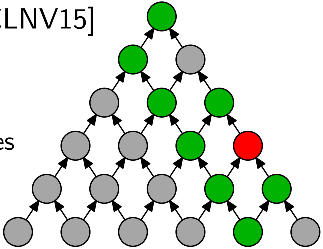

Lemma

∃ depth-r parallel decision tree for $Search(Peb_G)$ with $\leq c$ queries
$\Rightarrow$ Pebbler wins r-round Dymond–Tompa game on G in cost $\leq c + 1$.

Clinching the Argument

Prove round-cost trade-offs for Dymond–Tompa games on graphs G
(hacking graph constructions from [CS82, LT82, Nor12])

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Depth-query trade-offs for parallel decision trees for $\mathit{Search}\big(\mathit{Peb}_G\big)$

$\Downarrow$

Real communication round-cost trade-offs for $\mathit{Lift}\big(\mathit{Search}\big(\mathit{Peb}_G\big)\big)$

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Same communication lower bounds hold for $Search(Lift(Peb_G))$,
i.e., search problem for lifted formulas in [BHP10]

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Same communication lower bounds hold for $\mathit{Search}(\mathit{Lift}(\mathit{Peb}_G))$,
i.e., search problem for lifted formulas in [BHP10]

$\Downarrow$

Cutting planes length-space trade-off for $\mathit{Lift}(\mathit{Peb}_G)$

Some Remaining Open Questions

Communication complexity

- Smaller length of lift?
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Proof complexity

- Better Dymond–Tompa trade-offs?
- Reduction to black-white pebbling instead of Dymond–Tompa?
- Size-space trade-offs for Tseitin formulas à la [BBI12, BNT13]?
- Line space lower bounds for CP with bounded coefficients (strengthening [GPT15])

Take-Home Message

Summary of results

- Modern SAT solvers enormously successful in practice — key issue is to minimize time and memory consumption
- Modelled by proof size and space in proof complexity
- We show uniform trade-offs indicating that simultaneous optimization impossible for (essentially all) state-of-the-art techniques

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Thank you for your attention!

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