Certified CNF Translations for Pseudo-Boolean Solving

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SAT '22 paper joint with Stephan Gocht, Ruben Martins, and Andy Oertel

Pseudo-Boolean (PB) Solving

pseudo-Boolean formula:

$$x_1 + x_2 + x_3 \geq 2$$
$$x_2 + x_3 + x_4 \geq 2$$
$$\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 3$$

$\rightarrow$ Pseudo-Boolean solver $\rightarrow$ Result: SAT/UNSAT

- Input: Pseudo-Boolean formula (a.k.a. 0–1 integer linear program)
  - Collection of 0–1 integer linear constraints

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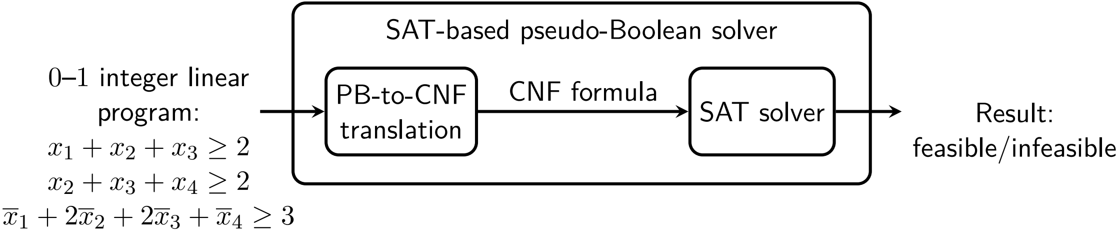
  - Native: SAT4J [LP10], ROUNDINGSAT [EN18]
  - SAT-based: MINISAT+ [ES06], OPEN-WBO [MML14], NAPS [SN15]

Pseudo-Boolean (PB) Solving



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Certifying SAT Solver Results with Proof Logging

SAT solver

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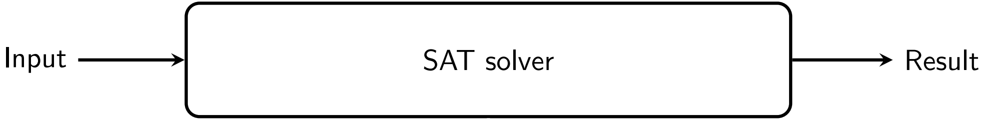

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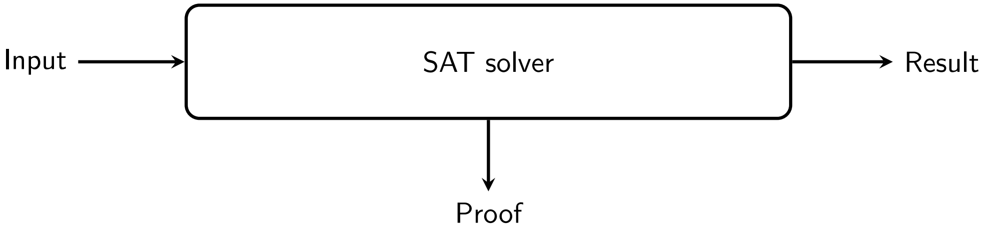

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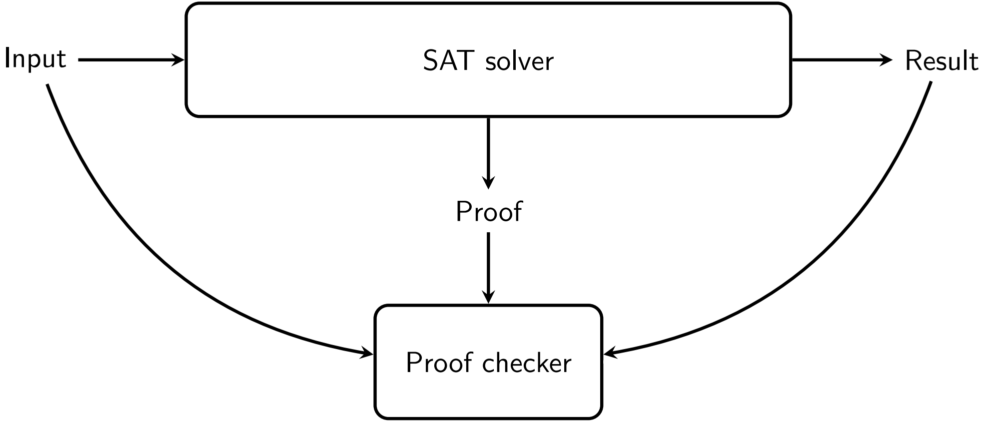

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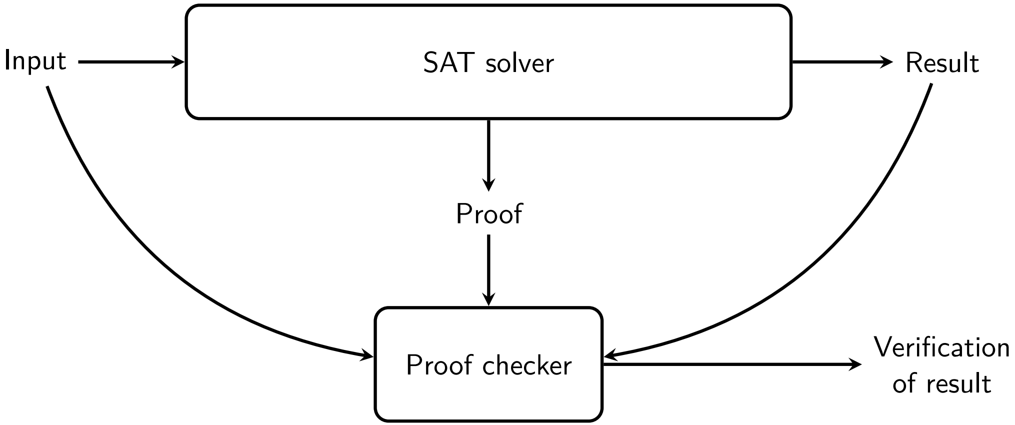

Certifying Pseudo-Boolean Solver Results with Proof Logging?

0–1 integer linear program:

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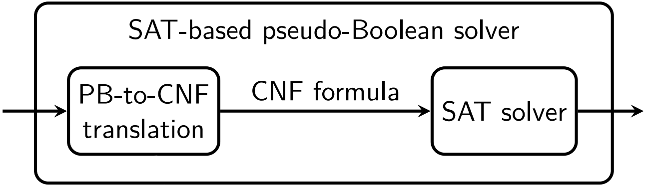
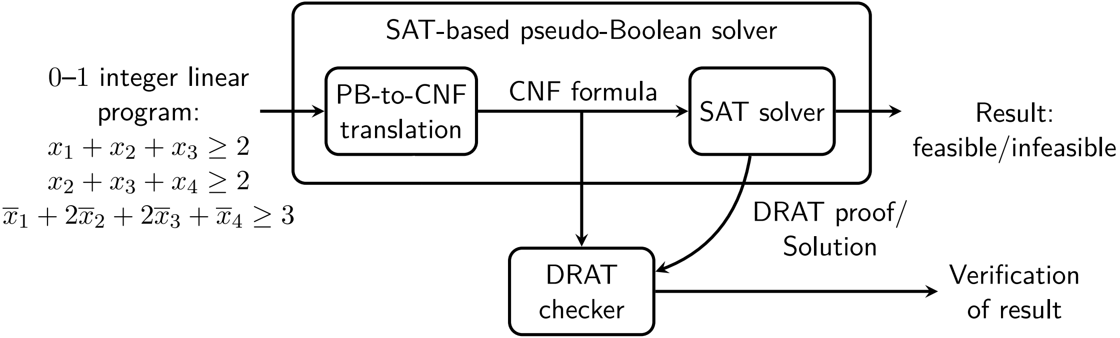


Result: feasible/infeasible

Certifying Pseudo-Boolean Solver Results with Proof Logging?



- Correctness of SAT solver result can be certified [HHW13a, HHW13b, WHH14]

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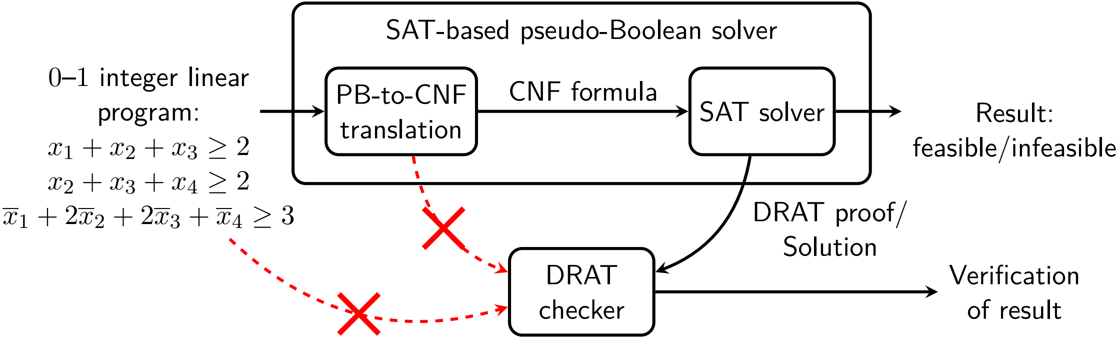

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- **PB-to-CNF translation uncertified!**

Example Pseudo-Boolean-to-CNF Translation

$\ell_1 + \ell_2 + \ell_3 \geq 2$

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$\ell_1 \lor \overline{s}_{1,1}$	$\ell_2 \lor \overline{s}_{2,2}$	$\overline{\ell}_3 \lor \overline{s}_{2,1} \lor s_{3,2}$	$\ell_3 \lor \overline{s}_{3,3}$
$\overline{\ell}_2 \lor s_{2,1}$	$s_{1,1} \lor \overline{s}_{2,2}$	$\overline{s}_{2,2} \lor s_{3,2}$	$s_{2,2} \lor \overline{s}_{3,3}$
$\overline{s}_{1,1} \lor s_{2,1}$	$\overline{\ell}_3 \lor s_{3,1}$	$\ell_3 \lor s_{2,2} \lor \overline{s}_{3,2}$	$s_{3,2}$
$\ell_2 \lor s_{1,1} \lor \overline{s}_{2,1}$	$\overline{s}_{2,1} \lor s_{3,1}$	$s_{2,1} \lor \overline{s}_{3,2}$	

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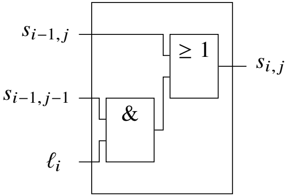

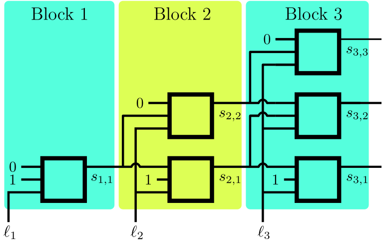

$$
\begin{array}{rrrr}
\bar\ell_1 \vee s_{1,1} & \bar\ell_2 \vee \bar s_{1,1} \vee s_{2,2} & \ell_3 \vee s_{2,1} \vee \bar s_{3,1} & \bar\ell_3 \vee \bar s_{2,2} \vee s_{3,3} \\
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Example Pseudo-Boolean-to-CNF Translation

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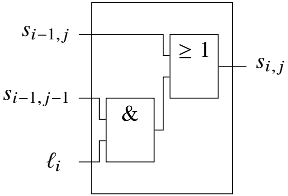
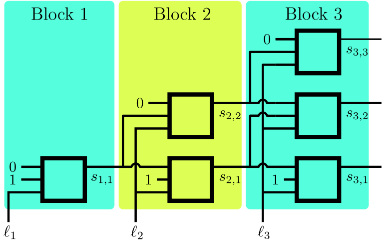
Meaning of $s_{i,j}$-variable:

$s_{i,j}$ true $\Leftrightarrow \ell_1 + \ldots + \ell_i \geq j$





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Pseudo-Boolean Proof Logging with VeriPB

- Inspired by SAT proof logging, but operates on 0–1 linear constraints
- Supports efficient proof logging for
  - SAT solving — including advanced techniques previously beyond efficient proof logging like
    - Gaussian elimination [GN21]
    - Symmetry breaking [BGMN23]
  - SAT-based optimization (MaxSAT) approaches like
    - Model-improving search [VDB22]
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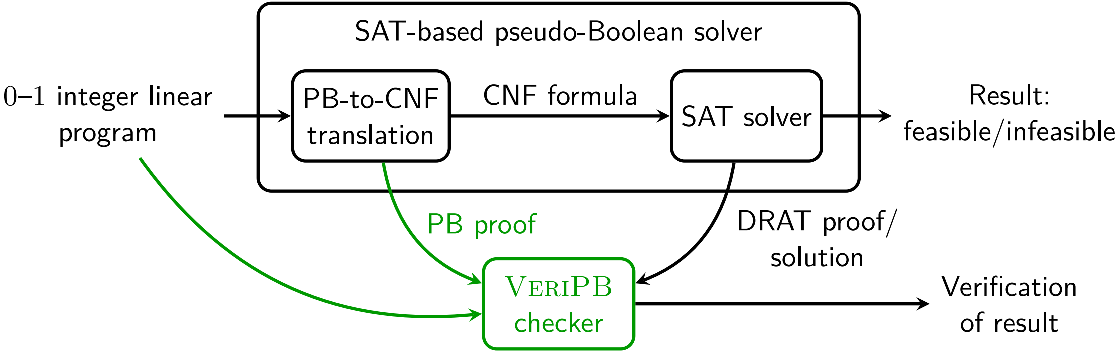
This work:

- Proof logging for translating pseudo-Boolean constraints to CNF
- General framework to certify many different encodings

End-to-End Verification for Pseudo-Boolean Solving



All the Technical Details on One Slide

All the Technical Details on One Slide (or Not)

How pretty much all PB-to-CNF translations work:

1. Design circuit evaluating left-hand side of 0–1 integer linear constraint
2. Encode circuit to CNF using Tseitin translation (one new variable per circuit wire)
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We solve this task by

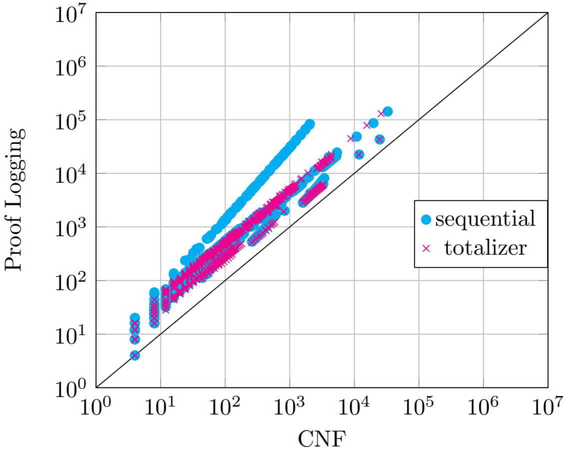
1. Understanding the circuit design
2. Deriving CNF encoding of circuit (easy)
3. Proving circuit output is true (tricky, but using pseudo-Boolean reasoning really helps)

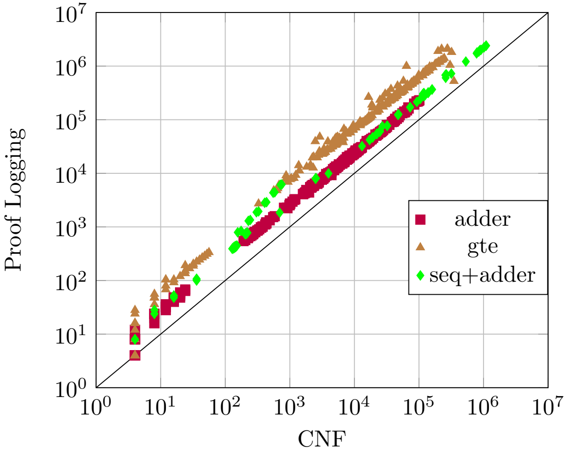
Experimental Evaluation

- Certified translations for CNF encodings with VeritasPBLib[1]
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proofs verified by proof checker VeriPB[2]
- Formulas solved with fork of Kissat[3] outputting VeriPB proofs
- Benchmarks from PB 2016 Evaluation[4] in 3 categories
  - Only cardinality constraints (sequential counter, totalizer)
  - Only general 0-1 ILP constraints (generalized totalizer, adder network)
  - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

[1] https://github.com/forge-lab/VeritasPBLib
[2] https://gitlab.com/MIAOresearch/software/VeriPB
[3] https://gitlab.com/MIAOresearch/tools-and-utilities/kissat_fork
[4] http://www.cril.univ-artois.fr/PB16/

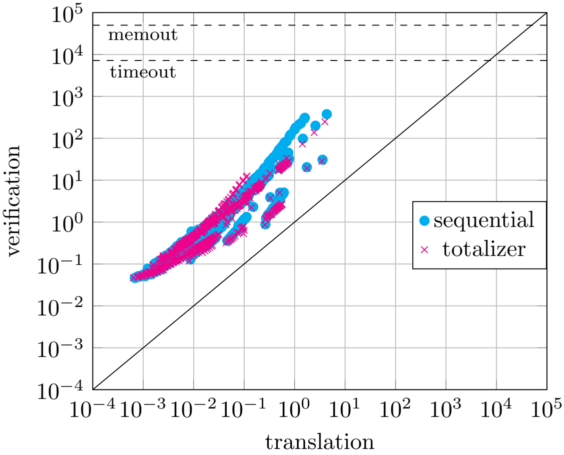
CNF Size vs Proof Size in KiB

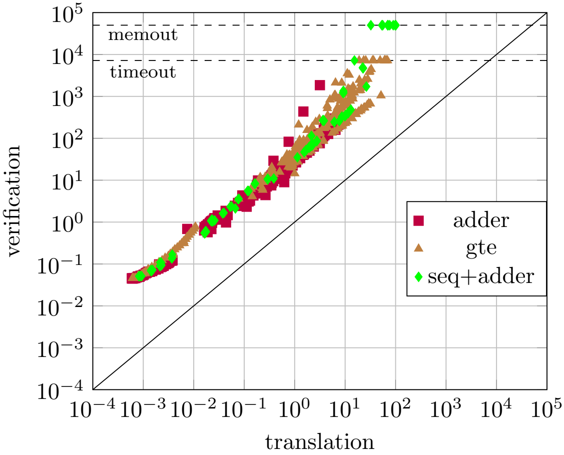




- Nice scaling for proof size in terms of original CNF formula size
- Except for some sequential encoding cases (which is not such a great encoding anyway)

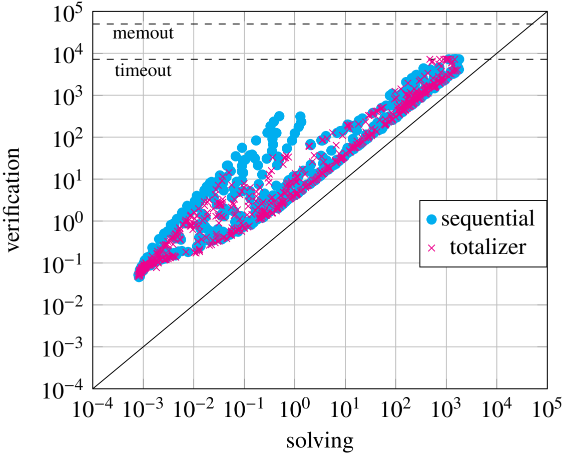
Translation Time vs Proof Checking Time in Seconds

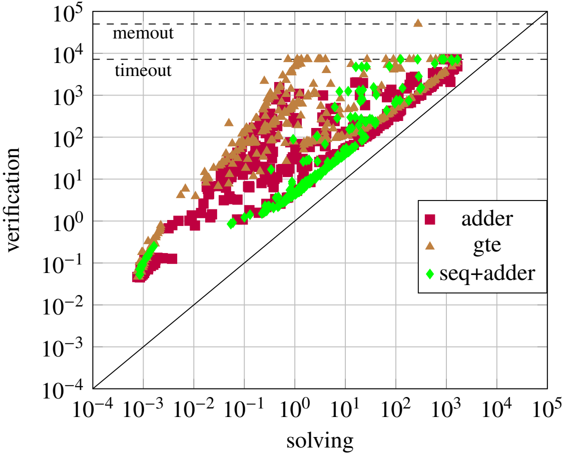




- Translation fast — only has to generate clauses and proof
- Proof checking slower — has to verify full proof

Solving Time vs Proof Checking Time in Seconds





- Room for improvement of end-to-end proof checking process
- But even first proof-of-concept implementation shows our approach is viable

Future Work

Improve performance and reliability:

- Cutting planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-TRIM [HHW13a])
- Fully formally verified proof checking (work in progress [BMM+23])

Extend proof logging further:

- PB-to-CNF translations with odd-even mergesort & bitonic sorting networks [Bat68]
- All of MaxSAT solving and (linear) pseudo-Boolean optimization
- Mixed integer linear programming
- Automated planning
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We're hiring! Talk to me to join the proof logging revolution! ☺

Take-Home Message

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- Boolean satisfiability (SAT) including advanced techniques [GN21, BGMN23]
- SAT-based optimization (MaxSAT) [VDB22, BBN+23]
- SAT-based pseudo-Boolean solving [**this work**]
- Constraint programming [EGMN20, GMN22, MM23]
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Action point: What can VeriPB do for you? ☺

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Thank you for your attention!

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