Pseudo-Boolean Solving: In Between SAT and ILP

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Theoretical Foundations of SAT/SMT Solving
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Pseudo-Boolean?

Pseudo-Boolean function: $f : \{0,1\}^n \to \mathbb{R}$

Studied since 1960s in operations research and 0-1 integer linear programming [BH02]

Restricted version: f represented as linear form [focus of this talk]

Many problems expressible as optimizing value of linear pseudo-Boolean function under linear pseudo-Boolean constraints

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See Simons boot camp tutorial `https://tinyurl.com/pbsolving` for (much) longer version of this talk

Pseudo-Boolean vs. SAT

- Pseudo-Boolean format richer than conjunctive normal form (CNF)

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
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- Yet close enough to SAT to benefit from SAT solving advances
- Also possible synergies with 0-1 integer linear programming (ILP)

Pseudo-Boolean Constraints and Normalized Form

In this talk, pseudo-Boolean constraints are 0-1 integer linear constraints

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Convenient to use normalized form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Some Types of Pseudo-Boolean Constraints

1. Clauses are pseudo-Boolean constraints

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

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4. Reified constraints encoding $z \Leftrightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

$$7\overline{z} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$
$$9z + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

Formulas, Decision Problems, and Optimization Problems

Pseudo-Boolean (PB) formula
Conjunction of pseudo-Boolean constraints
$F \doteq C_1 \wedge C_2 \wedge \cdots \wedge C_m$

Pseudo-Boolean Solving (PBS)
Decide whether F is satisfiable/feasible

Pseudo-Boolean Optimization (PBO)
Find satisfying assignment to F that minimizes objective function $\sum_i w_i \ell_i$
(Maximization: minimize $-\sum_i w_i \ell_i$)

Approaches for Pseudo-Boolean Problems

1. Pseudo-Boolean (PB) solving and optimization [main focus]
2. MaxSAT solving
3. Integer linear programming (ILP) — or, more generally, mixed integer linear programming (MIP)

Two Approaches to Pseudo-Boolean Solving

Re-encode to CNF and run conflict-driven clause learning (CDCL)

- MiniSat+ [ES06]
- Open-WBO [MML14]
- NaPS [SN15]

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Native reasoning with pseudo-Boolean constraints

- PRS [DG02]
- Galena [CK05]
- Pueblo [SS06]
- Sat4j [LP10]
- RoundingSat [EN18]

Performance of CDCL-Based Pseudo-Boolean Solving

- CDCL-based pseudo-Boolean can be very competitive (sometimes beating native pseudo-Boolean solvers hands down)
- Extension variables potentially gives solver lots of power
  - Allows branching over complex statements
  - Can learn clauses corresponding to polytopes in original problem
- But performance gain from extension variables seems quite sensitive to input order [EGNV18]
- And sometimes extension variables cannot make up for CDCL being exponentially weaker than pseudo-Boolean reasoning [EGNV18]

Some Research Questions

1. How to find best CNF encodings of PB constraints for given problem?
   - Trade-offs between propagation strength and encoding size?
   - Rigorous mathematical insights?

2. How do CDCL-based and "native" cutting-planes-based PB solving approaches compare?
   - Theoretical results on computational complexity?
   - Harness complementary strengths in applied solvers?

"Native" Pseudo-Boolean Conflict-Driven Search

Want to do "same thing" as in conflict-driven clause learning (CDCL) SAT solving [MS96, BS97, MMZ⁺01]

But with cutting planes reasoning on PB constraints without re-encoding

- Variable assignments
  1. Always propagate forced assignment if possible
  2. Otherwise make assignment using decision heuristic
- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to constraint database and backjump

The Cutting Planes Proof System [CCT87, CK05]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

Saturation $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min\{a_i, A\} \cdot \ell_i \geq A}$$

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Division by 2 $\dfrac{x + 2y + 4z \geq 3}{x + y + 2z \geq 2}$ Saturation $\dfrac{x + 2y + 4z \geq 3}{x + 2y + 3z \geq 3}$

Some PB Solving Challenges I: Input Format

1. CNF: PB solvers degenerate to CDCL for CNF inputs — how to harness power of cutting planes in this setting?
   - Cardinality constraint detection proposed as preprocessing [BLLM14] or inprocessing [EN20]
   - Not yet competitive in practice
2. Linear programming: Sometimes very poor performance even on infeasible 0-1 LPs!
   - Unclear why
   - Very easy for cutting planes in theory
3. Preprocessing/presolving: Important in SAT solving and integer linear programming, but not done in PB solvers — why?
   - Follow up on preliminary work on PB preprocessing in [MLM09]?
   - Use presolver PaPILO [PaP] from MIP solver SCIP [SCI]?

Some PB Solving Challenges II: Conflict Analysis

1. Many more degrees of freedom than in CDCL, e.g.:
   - Choice of Boolean rule (division, saturation, or combination?)
   - Learn general PB constraints or more limited form?
   - How far to backjump when learned constraint asserting at many levels?
   - How large precision to use in integer arithmetic?
2. How to assess quality of learned constraints?
3. Theoretical potential and limitations poorly understood [VEG$^+$18]
   - Separations of subsystems of cutting planes?
   - In particular, is division reasoning stronger than saturation? [GNY19]

Linear Search SAT-UNSAT (LSU) Algorithm

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\rho_{\text{best}} = \emptyset$ and repeat the following:

1. Run SAT/PB solver
2. If solver returns `UNSATISFIABLE`, output ρ_{best} and terminate
3. Otherwise, let $\rho_{\text{best}} :=$ returned solution ρ
4. Add constraint $\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
5. Start over from the top

More on Linear Search

Properties of linear search SAT-UNSAT:

- Can get **some decent** solution quickly, even if not optimal one
- Important for anytime solving (when time is limited and something is better than nothing)
- But get no estimate of how good the solution is

Core-Guided Pseudo-Boolean Search

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Core-guided PB search: assume optimistically that objective can reach best imaginable value; derive contradiction if not possible

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Set $val_{\text{best}} = 0$ and repeat the following:

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5. Start over from top (with modified objective function)

Properties of (Pure) Core-Guided Search

- Can get decent lower bounds on solution quickly
- Helps to cut off parts of search space that are "too good to be true"
- But find no actual solution until the final, optimal one
- Also, no estimate of how good the lower bound is
- Linear search much better at finding solutions — so try to get the best of both worlds by combining the two!

Evaluation of Core-Guided PB Solver in [DGD$^+$21]

RoundingSat variants with core-guided (CG) and linear search (LSU)

#instances solved to optimality; highlighting **1st**, **2nd**, and **3rd** best

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	PB16opt (1600)	MIPopt (291)	KNAP (783)	CRAFT (985)
Hybrid (interleave CG & LSU)	**968**	**78**	306	**639**
HybridCl (w/ clausal cores)	937	75	298	**618**
HybridNL (w/ non-lazy variables)	936	70	186	607
HybridClNL (w/ both)	917	67	203	612
RoundingSat (only LSU)	853	75	341	309
CoreGuided (only CG)	911	61	43	595
CoreBoosted (10% CG, then LSU)	**959**	**80**	**344**	580
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Significant improvement over PB state of the art, but MIP still better

Mixed Integer Linear Programming

Mixed integer linear program

- Minimize $\sum_j a_j x_j$
- Subject to $\sum_j a_{i,j} x_j \leq A_i$, $i = 1, \ldots, m$
- $x_j \in \mathbb{N}$ for $j = 1, \ldots, n$
- $x_j \in \mathbb{R}_{\geq 0}$ for $j = n+1, \ldots, N$

- Linear constraints
- Integer-valued variables
- Real-valued variables
- Linear objective function

- No real-valued variables: integer linear program (ILP)
- $0 \leq x_j \leq 1$ for all j: 0-1 ILP
- Vacuous objective $\sum_j 0 \cdot x_j$: decision problem
- But MIP best for optimization

MIP Solving at a High Level

1. Preprocessing (called presolving)
2. Linear programming relaxations + branch-and-bound
3. Add cutting planes ruling out infeasible LP-solutions (branch-and-cut method going back to [Gom58])
4. Heuristics for quickly finding good feasible solutions

Combining PB Solving and Mixed Integer Programming

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- Rich variety of cut generation routines
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Why not merge the two to get the best of both worlds of SAT-style conflict-driven search and MIP-style branch-and-cut?

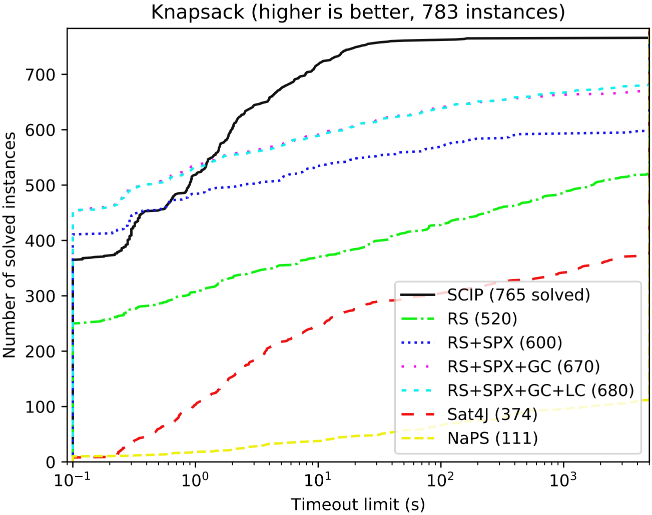
Experimental Results for Knapsack Benchmarks [Pis05]

ROUNDINGSAT (RS) enhanced with

- LP solver SOPLEX (SPX) (from SCIP)
- Gomory cuts (GC)
- shared learned PB cuts (LC)

as in [DGN21]

compared to other solvers



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PB16opt (1600)	**1057**	862	**988**	986	**993**	776	896
MIPdec (556)	**264**	203	**263**	**261**	259	169	170
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Hybrid PB-LP solver well-rounded — always competitive with best solver

Some Future Research Directions for PB-LP Integration

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2. Make more intelligent use in PB solver of information from solutions to LP relaxations
3. Use MIP presolving in pseudo-Boolean solvers

Some Future Research Directions for PB-LP Integration

1. Fine-tune heuristics
   - Improved LP-based cut generation?
   - Smarter sharing of PB constraints with LP solver?
   - Dynamic allocation of PB and LP solving time based on contributions?
2. Make more intelligent use in PB solver of information from solutions to LP relaxations
3. Use MIP presolving in pseudo-Boolean solvers
4. Use MIP cut rules to improve pseudo-Boolean conflict analysis

Balancing the Picture

Cutting-planes-based pseudo-Boolean solvers sometimes outperform even commercial MIP solvers by orders of magnitude:

- Arithmetic circuit verification [LBD^{+}20]
- Matching of children with adoptive families (compared to [DGG^{+}19])
- Automated planning using neural networks (compared to [SS18], see also [SDNS20] — reified constraints hard for MIP)

Summing up

- Pseudo-Boolean optimization powerful and expressive framework
- Can be attacked with methods from
  - SAT solving and MaxSAT solving
  - "Native" cutting-planes-based pseudo-Boolean reasoning
  - Mixed integer linear programming
- Approaches with complementary strengths — room for synergies?
- For cutting-planes-based reasoning, challenges regarding
  - Algorithm design
  - Efficient implementation
  - Theoretical understanding
- But cutting-planes-based solvers sometimes very powerful — worth trying out if you have a MaxSAT/PB optimization problem!

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Thank you for your attention!

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