A (Biased) Proof Complexity Survey for SAT Practitioners

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17th International Conference on Theory and Applications of Satisfiability Testing
Vienna, Austria
July 14–17, 2014

Proof Complexity and SAT Solving

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- SAT proven NP-complete by Stephen Cook in 1971
- Hence totally intractable in worst case (probably)
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- State-of-the-art solvers deal with millions of variables
- But best solvers still based on methods from early 60s
- Tiny formulas known that are totally beyond reach

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What can proof complexity say about SAT solving?

Focus of This Survey

Proof systems behind some current approaches to SAT solving:

- Conflict-driven clause learning — resolution
- Gröbner basis computations — polynomial calculus
- Pseudo-Boolean solvers — cutting planes

Survey (some of) what is known about these proof systems

Show some of the "benchmark formulas" used

By necessity, selective and somewhat subjective coverage — apologies in advance for omissions

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- Mostly assume formulas k-CNFs (for simplicity of exposition)
  Conversion to 3-CNF (most often) doesn't change much
- N denotes size of formula (# literals, which is $\approx$ # clauses)

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

Refutation ends when empty clause $\perp$ derived

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Can represent refutation as

- annotated list or
- DAG

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2,4)$
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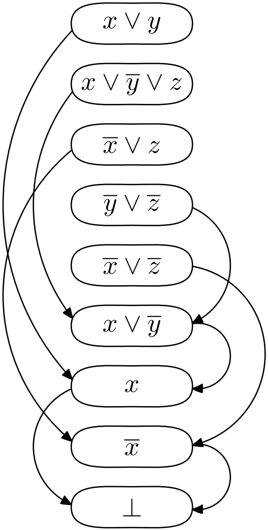

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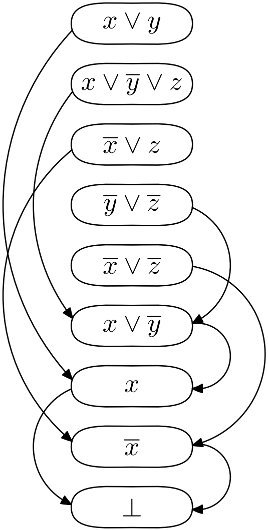
Tree-like resolution if DAG is tree



Resolution Size/Length

Size/length $= \#$ clauses in refutation

Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Examples of Hard Formulas w.r.t Resolution Length (1/3)

Pigeonhole principle (PHP) [Hak85]

"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
$p_{1,j} \lor p_{2,j} \lor \cdots \lor p_{n+1,j}$	every hole j gets a pigeon

Even onto functional PHP formula is hard for resolution

But only length lower bound $\exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ in terms of formula size

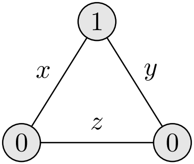
Examples of Hard Formulas w.r.t Resolution Length (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of edges around vertex = label

Requires length $\exp(\Omega(N))$ on well-connected so-called expanders



$$
\begin{array}{llll}
 & (x \lor y) & \land & (\overline{x} \lor z) \\
\land & (\overline{x} \lor \overline{y}) & \land & (y \lor \overline{z}) \\
\land & (x \lor \overline{z}) & \land & (\overline{y} \lor z)
\end{array}
$$

Examples of Hard Formulas w.r.t Resolution Length (3/3)

Random k-CNF formulas [CS88]
Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

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And more...

- k-colourability [BCMM05]
- Independent sets and vertex covers [BIS07]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera...

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Width lower bound $\Rightarrow$ length lower bound

Much less obvious. . .

Width Lower Bounds Imply Length Lower Bounds

Theorem ([BW01])

$$\mathit{length} \geq \exp\left(\Omega\left(\frac{\mathit{width}^2}{\mathit{formula\ size}\ N}\right)\right)$$

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- Almost all known lower bounds on length derivable via width

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For tree-like resolution have length $\geq 2^{\text{width}}$ [BW01]

General resolution: width up to $\mathcal{O}\left(\sqrt{N \log N}\right)$ implies no length lower bounds — possible to tighten analysis? **No!**

Optimality of the Length-Width Lower Bound

Ordering principles [Stå96, BG01]
"Every (partially) ordered set $\{e_1, \ldots, e_n\}$ has minimal element"

Variables $x_{i,j} =$ "$e_i < e_j$"

$\overline{x}_{i,j} \lor \overline{x}_{j,i}$	anti-symmetry; not both $e_i < e_j$ and $e_j < e_i$
$\overline{x}_{i,j} \lor \overline{x}_{j,k} \lor x_{i,k}$	transitivity; $e_i < e_j$ and $e_j < e_k$ implies $e_i < e_k$
$\bigvee_{1 \leq i \leq n,\, i \neq j} x_{i,j}$	e_j is not a minimal element

Can also add "total order" axioms

$x_{i,j} \lor x_{j,i}$	totality; either $e_i < e_j$ or $e_j < e_i$

Reuftable in resolution in length $\mathcal{O}(N)$
Requires resolution width $\Omega(\sqrt[3]{N})$ (3-CNF version)

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by SAT solver memory usage (but also intrinsically interesting for proof complexity)

Can be measured in different ways — focus here on most common measure clause space

Space at step t: # clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \vee y$	Axiom
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Example: Space at step 7 . . .

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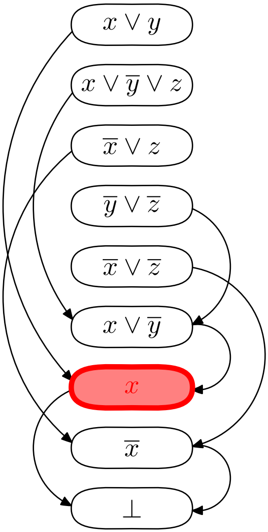

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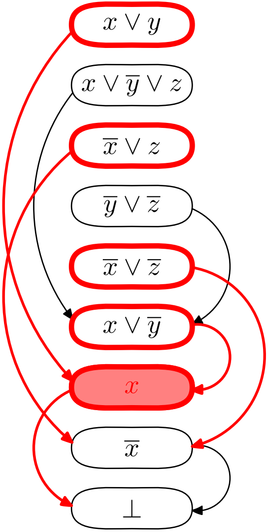
Example: Space at step 7 is 5



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ [ET01]

Lower bounds for

- Pigeonhole principle [ABRW02, ET01]
- Tseitin formulas [ABRW02, ET01]
- Random k-CNFs [BG03]

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Results always matching width bounds

And proofs of very similar flavour... What is going on?

Space vs. Width

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Are space and width asymptotically always the same? **No!**

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Pebbling formulas [BN08]

- Can be refuted in width $\mathcal{O}(1)$
- May require space $\Omega(N/\log N)$

A bit more involved to describe than previous benchmarks...

Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
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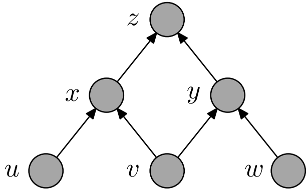

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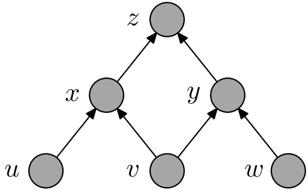

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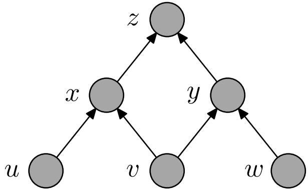

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Pebbling Formulas: Vanilla Version

CNF formulas encoding so-called pebble games on DAGs

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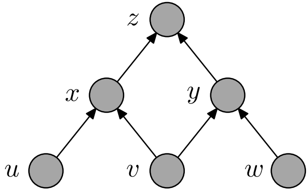

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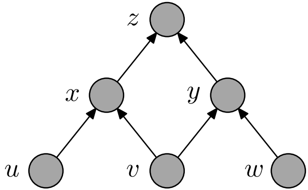

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Have been useful in proof complexity before in various contexts

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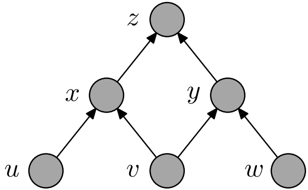

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Have been useful in proof complexity before in various contexts

Hope that pebbling properties of DAG somehow carry over to resolution refutations of pebbling formulas. **Except...**

Substituted Pebbling Formulas

Won't work — solved by unit propagation, so supereasy

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Now CNF formula inherits pebbling graph properties!

Space-Width Trade-offs

Given a formula easy w.r.t. these complexity measures, can refutations be optimized for two or more measures?

For space vs. width, the answer is a strong no

Theorem ([Ben09])

There are formulas for which

- *exist refutations in width $\mathcal{O}(1)$*
- *exist refutations in space $\mathcal{O}(1)$*
- *optimization of one measure causes (essentially) worst-case behaviour for other measure*

Holds for vanilla version of pebbling formulas

Length-Space Trade-offs

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure* causes *dramatic blow-up for other measure*

Holds for

- Substituted pebbling formulas over the right graphs
- Tseitin formulas over long, narrow rectangular grids

So no meaningful simultaneous optimization possible for length and space in the worst case

Length-Width Trade-offs?

What about length versus width?

[BW01] transforms short refutation to narrow one, but blows up length exponentially

- Is this blow-up inherent?
- Or just an artifact of the proof?

Open Problem

Are there length-width trade-offs in resolution? Or is a narrow refutation never much longer than the shortest one?

Recap of Complexity Measures for Resolution

Recall that N = size of formula

Length

clauses in refutation — at most $\exp(N)$

Width

Size of largest clause in refutation — at most N

Space

Max # clauses one needs to remember when "verifying correctness of refutation" — at most N (!)

Proof Complexity Measures and CDCL Hardness

Recall $\log(\text{length}) \lesssim \text{width} \lesssim \text{space}$

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- CDCL polynomially simulates resolution [PD11]
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Space

- In practice, memory consumption important bottleneck
- Space complexity gives lower bound on clause database size
- Plus assumes solver knows **exactly** which clauses to keep $\Rightarrow$ in reality, probably (much) more memory needed

Relations Between Theoretical and Practical Hardness?

1. Are width or even space lower bounds relevant indicators of CDCL hardness?
2. Or is it true in practice that CDCL does essentially as well as resolution w.r.t. length/running time?
3. Can CDCL even do as well as resolution w.r.t. time and space simultaneously?

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Not mathematically well-defined questions...

But perhaps still possible to perform experiments and draw interesting conclusions?

Practical Experimental Evaluation

- Proposed by [ABLM08]
- First(?) systematic attempt in [JMNŽ12]
- Length as a proxy for hardness seems too optimistic. . .
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- Is running time essentially the same?
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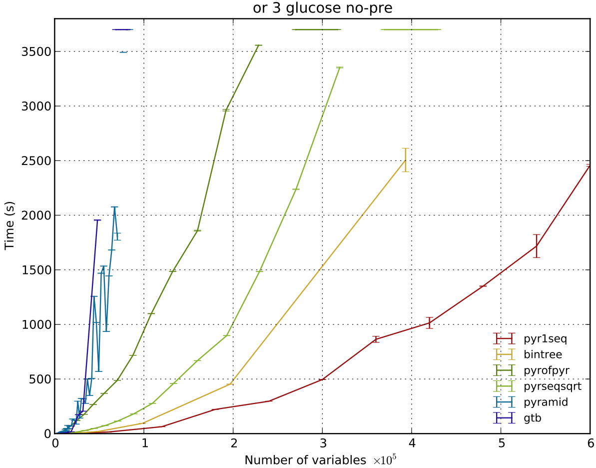
But sometimes they **really** don't...

(*) Note: such formulas nontrivial to find; only know one construction

Example Results for Glucose Without Preprocessing



Looks nice. . . "Easy" formulas solved fast; "hard" take longer time

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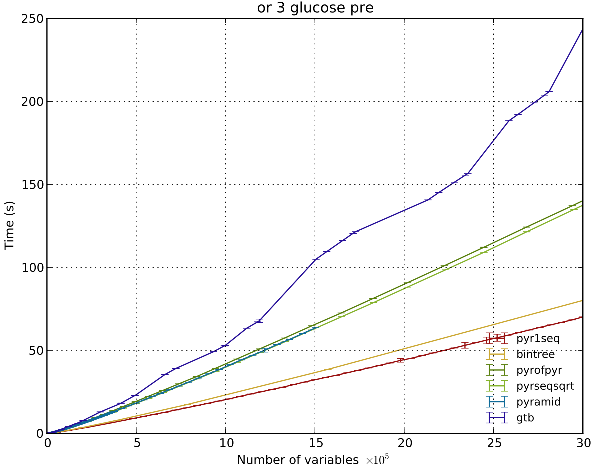
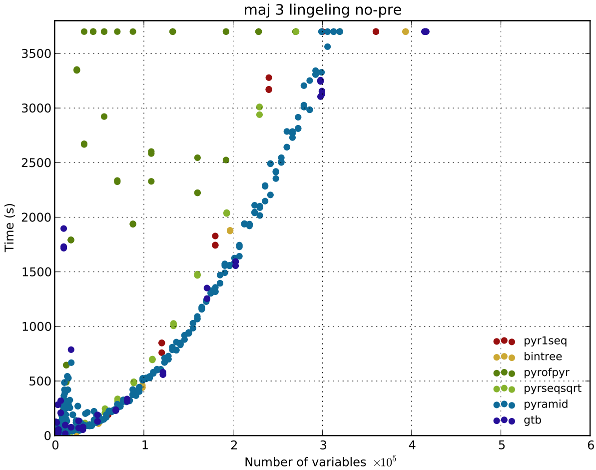


Preprocessing makes formulas much easier, but this still looks nice

Some Lingeling Results (Without Preprocessing)



But sometimes we see pretty random behaviour. . .

Practical Conclusions?

- No firm conclusions — both space and width seem relevant
- And sometimes other structural properties more important?
- More generally, CDCL performance on combinatorial benchmarks sometimes surprising; e.g.:
  - For PHP, worse behaviour with heuristics than without
  - For ordering principles, highly dependent on specific solver
  - Sometimes "easy" formulas harder than "hard" ones?! [MN14]

Open Problems

- *Could explanations of above phenomena help us understand CDCL better?*
- *Could controlled experiments on easily scalable theoretical benchmarks yield other interesting insights?*

Polynomial Calculus (or Actually PCR)

Introduced in [CEI96]; below modified version from [ABRW02]

Clauses interpreted as polynomial equations over finite field
Any field in theory; $\mathrm{GF}(2)$ in practice
Example: $x \lor y \lor \overline{z}$ gets translated to $xy\overline{z} = 0$
(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

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(Think of $0 \equiv \mathit{true}$ and $1 \equiv \mathit{false}$)

Derivation rules

Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + \overline{x} = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

Goal: Derive $1 = 0 \Leftrightarrow$ no common root $\Leftrightarrow$ formula unsatisfiable

Size, Degree and Space

Write out all polynomials as sums of monomials
W.l.o.g. all polynomials multilinear (because of Boolean axioms)

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W.l.o.g. all polynomials multilinear (because of Boolean axioms)

Size — analogue of resolution length
total # monomials in refutation (counted with repetitions)
Can also define length measure — might be much smaller

Degree — analogue of resolution width
largest degree of monomial in refutation

(Monomial) space — analogue of resolution (clause) space
max # monomials in memory during refutation (with repetitions)

Polynomial Calculus Simulates Resolution

Polynomial calculus can simulate resolution proofs efficiently with respect to length/size, width/degree, and space simultaneously

- Can mimic resolution refutation step by step
- Hence worst-case upper bounds for resolution carry over

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Example: Resolution step:

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation:

$$\dfrac{x\overline{y}z = 0 \qquad \dfrac{\dfrac{\overline{yz} = 0}{x\overline{yz} = 0} \qquad \dfrac{\dfrac{z + \overline{z} - 1 = 0}{\overline{y}z + \overline{yz} - \overline{y} = 0}}{x\overline{y}z + x\overline{yz} - x\overline{y} = 0}}{-x\overline{y}z + x\overline{y} = 0}}{x\overline{y} = 0}$$

Polynomial Calculus Strictly Stronger than Resolution

Polynomial calculus strictly stronger w.r.t. size and degree

- Tseitin formulas on expanders (just do Gaussian elimination)
- Onto functional pigeonhole principle [Rii93]

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Open Problem

Show that polynomial calculus is strictly stronger than resolution w.r.t. space

Size vs. Degree

- Degree upper bound ⇒ size upper bound [CEI96]
  Qualitatively similar to resolution bound
  A bit more involved argument
  Again essentially tight by [ALN14]

- Degree lower bound ⇒ size lower bound [IPS99]
  Precursor of [BW01] — can do same proof to get same bound

- Size-degree lower bound essentially optimal [GL10]
  Example: again ordering principle formulas

- Most size lower bounds for polynomial calculus derived via degree lower bounds (but machinery less developed)

Examples of Hard Formulas w.r.t. Size (and Degree)

Pigeonhole principle formulas
Follows from [AR03]
Earlier work on other encodings in [Raz98, IPS99]

Tseitin formulas with "wrong modulus"
Can define Tseitin-like formulas counting $\bmod\ p$ for $p \neq 2$
Hard if $p \neq$ characteristic of field [BGIP01]

Random k-CNF formulas
Hard in all characteristics except 2 [BI99]
Lower bound for all characteristics in [AR03]

Bounds on Polynomial Calculus Space

Lower bound for PHP with wide clauses [ABRW02]

k-CNFs much trickier — sequence of lower bounds for

- Obfuscated 4-CNF versions of PHP [FLN$^+$12]
- Random 4-CNFs [BG13]
- Tseitin formulas on (some) expanders [FLM$^+$13]

Open Problems

- *Prove tight space lower bounds for Tseitin on any expander*
- *Prove any space lower bound on random 3-CNFs*
- *Prove any space lower bound for any 3-CNF!?*

Space vs. Degree

Open Problem (analogue of [AD08])

Is it true that space $\geq$ degree $+ \mathcal{O}(1)$?

Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM+13]

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Partial progress: if formula requires large resolution width, then XOR-substituted version requires large space [FLM⁺13]

Optimal separation of space and degree in [FLM⁺13] by flavour of Tseitin formulas which

- can be refuted in degree $\mathcal{O}(1)$
- require space $\Omega(N)$
- but separating formulas depend on characteristic of field

Open Problem

Prove space lower bounds for substituted pebbling formulas (would give space-degree separation independent of characteristic)

Trade-offs for Polynomial Calculus

- Strong, essentially optimal space-degree trade-offs [BNT13]
  Same vanilla pebbling formulas as for resolution
  Same parameters

- Strong size-space trade-offs [BNT13]
  Same formulas as for resolution
  Some loss in parameters

Open Problem

Are there size-degree trade-offs in polynomial calculus?

Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...

Algebraic SAT Solvers?

- Quite some excitement about Gröbner basis approach to SAT solving after [CEI96]
- Promise of performance improvement failed to deliver
- Meanwhile: the CDCL revolution...
- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus
- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?
- Some shortcut seems to be needed — full Gröbner basis computation does too much work

Cutting Planes

Introduced in [CCT87]

Clauses interpreted as linear inequalities over the reals with integer coefficients

Example: $x \vee y \vee \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

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Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

No (useful) analogue of width/degree

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Cutting planes

- simulates resolution efficiently w.r.t. length/size and space simultaneously
- is strictly stronger w.r.t. length/size — can refute PHP efficiently [CCT87]

Open Problem

Show cutting planes strictly stronger than resolution w.r.t. space

Hard Formulas w.r.t Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with a k-clique is not $(k-1)$-colourable"

Lower bound via interpolation and circuit complexity

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Lower bound via interpolation and circuit complexity

Open Problems

Prove length lower bounds for cutting planes

- *for Tseitin formulas*
- *for random k-CNFs*
- *for any formula using other technique than interpolation*

Hard Formulas w.r.t Cutting Planes Space?

No space lower bounds known except conditional ones:

- Short cutting planes refutations of Tseitin formulas on expanders require large space [GP14]
  (But such short refutations probably don't exist anyway)

- Short cutting planes refutations of (some) pebbling formulas require large space [HN12, GP14]
  (and such short refutations do exist; hard to see how exponential length could help bring down space)

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Above results obtained via communication complexity

No (true) length-space trade-off results known
(Although results above can also be phrased as trade-offs)

Geometric SAT Solvers?

- Some work on pseudo-Boolean solvers using (subset of) cutting planes
- Seems hard to make competitive with CDCL on CNFs
- One key problem to recover cardinality constraints
- **But...** If cardinality constraints can be detected, then solvers can do really well (at least on combinatorial benchmarks)
- E.g., PHP formulas and also zero-one design formulas become easy [BBLM14]

Building SAT Solvers on Extended Resolution?

- Resolution + introduce new variables to name subformulas
- Without restrictions, corresponds to extended Frege
- Extremely strong — pretty much no lower bounds known
- In order to study extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Summing up

- Overview of resolution, polynomial calculus and cutting planes (More details in conference proceedings or survey [Nor13])
- Resolution fairly well understood
- Polynomial calculus less so
- Cutting planes almost not at all
- Could there be interesting connections between proof complexity measures and hardness of SAT?
- How can we build efficient SAT solvers on stronger proof systems than resolution?

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Thank you for your attention!

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