On the Virtue of Succinct Proofs: Amplifying Communication Complexity Hardness to Time-Space Trade-offs in Proof Complexity

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Joint work with Trinh Huynh

The SAT Problem in Theory and Practice

- SAT NP-complete and so probably intractable in worst case
- But enormous progress on applied algorithms last 10-15 years
- Surprising fact 1: State-of-the-art SAT solvers can deal with real-world instances containing millions of variables
- Surprising fact 2: Best SAT solvers today still based on methods from early 1960s
- Algebraic and geometric methods more efficient in theory but **not** so far in practice

SAT Solving and Proof Complexity

SAT solving

- Constructive (almost deterministic) algorithms
- Key resources for solvers: time and memory
- Ideally minimize simultaneously

Proof complexity

- Study proofs, i.e., nondeterministic algorithms
- Complexity measures: proof size and proof space
- Lower bounds for optimal algorithms

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Hope to understand potential and limitation of SAT solvers by studying corresponding proof systems

Complexity measures also natural and interesting in their own right

This talk: Size-space trade-offs for algebraic and geometric systems

Some Terminology and Notation

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: all clauses of size $\leq k$ (some constant)
- Goal: Refute given CNF formula (i.e., prove it is unsatisfiable)
- All formulas in this talk are k-CNFs
  (cleanest and most interesting case)

The Theoretical Model

- Proof system operates with lines of some syntactic form
- Proof/refutation is "presented on blackboard"
- Derivation steps:
  - Write down axiom clauses of CNF formula being refuted (as encoded by proof system)
  - Infer new lines by deductive rules of proof system
  - Erase lines not currently needed (to save space on blackboard)
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Complexity Measures: Length, Size and Space

Length
\# derivation steps

Size
$\approx$ total # symbols in proof counted with repetitions

Space
$\approx$ max size of blackboard to carry out proof
(e.g., space 3 for this blackboard)

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Note that:

1. These are (very) informal definitions only — see paper for details
2. Length and size can be very different but we won't distinguish between them here

Resolution

Basis for the most successful SAT solvers to date
(DPLL method plus clause learning)

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- Optimal (exponential) lower bounds on size [Urquhart '87; Chvátal & Szemerédi '88]
- Optimal (linear) lower bounds on clause space [Torán '99; Alekhnovich, Ben-Sasson, Razborov & Wigderson '00]
- Strong size-space trade-offs [Ben-Sasson & N. '11; Beame, Beck & Impagliazzo '12]

Polynomial Calculus (or Actually PCR [ABRW '00])

Clauses interpreted as polynomial equations over finite field

E.g., $x \lor y \lor \overline{z}$ translated to $x'y'z = 0$

Show no common root by deriving $1 = 0$

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Boolean axioms $\dfrac{}{x^2 - x = 0}$

Negation $\dfrac{}{x + x' = 1}$

Linear combination $\dfrac{p = 0 \qquad q = 0}{\alpha p + \beta q = 0}$

Multiplication $\dfrac{p = 0}{xp = 0}$

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- Optimal (exponential) lower bounds on size
  [Alekhnovich-Razborov '01] and others
- Only recently lower bounds on monomial space for k-CNFs
  [Filmus, Lauria, N., Thapen & Zewi '12] building on [ABRW '00]
  But not optimal(!?)
- No size-space trade-offs

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Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

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- Only one (exponential) lower bounds on size [Pudlák '97]
- No lower bounds on line space
- No size-space trade-offs

Trade-offs for Polynomial Calculus and Cutting Planes

We make some progress on understanding space and size-space trade-offs in polynomial calculus and cutting planes

Theorem (Informal)

There are k-CNF formulas $\{F_n\}_{n=1}^{\infty}$ of size $\Theta(n)$ such that

- *resolution can refute F_n in length $\mathcal{O}(n)$ (and hence so can polynomial calculus and cutting planes)*
- *any polynomial calculus or cutting planes refutation of F_n in length L and space s must have*

$$s \log L \gtrsim \sqrt[4]{n}$$

Proof Ingredients

- Pebbling
- Communication complexity
- Lifting

Pebbling Formulas

CNF formulas encoding pebble games played on DAGs (as studied in 1970s and 1980s)

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
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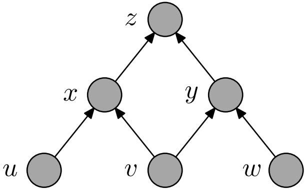

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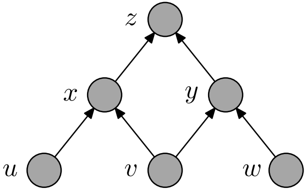

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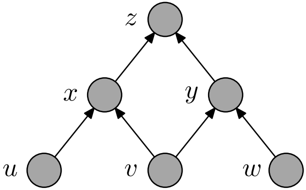

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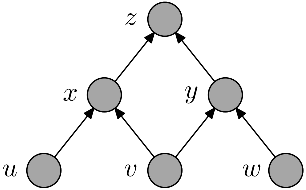

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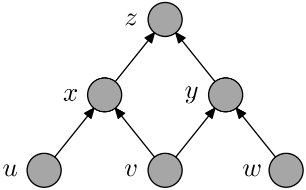

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Appeared in various contexts in [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and other papers

Used to study size and space in resolution in [N. '06, N. & Håstad '08, Ben-Sasson & N. '08, '11]

Two-Player Randomized Communication Complexity

- Alice has private input x and private source of randomness
- Bob has private input y and private source of randomness
- Both have unbounded computational powers
- Want to compute $f(x, y)$ by sending messages back and forth
- Output correct for any x and y except with error probability ε
- Communication cost: max # bits communicated on any x and y

Falsified Clause Search Problem

Fix:

- unsatisfiable CNF formula F
- (devious) partition of $Vars(F)$ between Alice and Bob

Falsified clause search problem $Search(F)$

Input: Assignment α to $Vars(F)$ split between Alice and Bob

Output: Clause $C \in F$ such that $\alpha(C) = 0$

Actually, computing not function but relation — more about that later

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Evaluate blackboard configurations of a refutation of F under α

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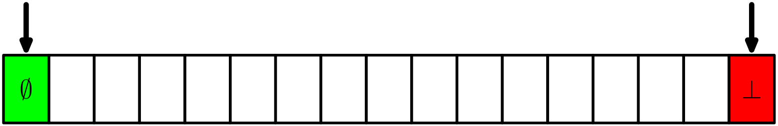

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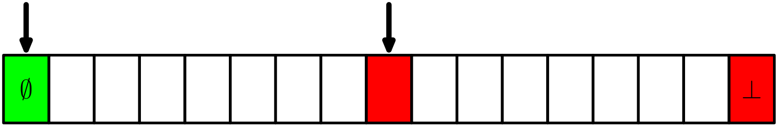

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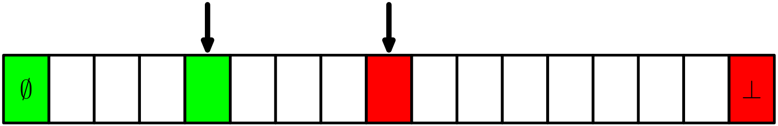

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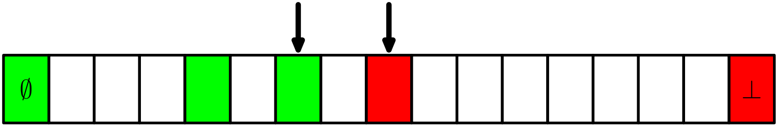

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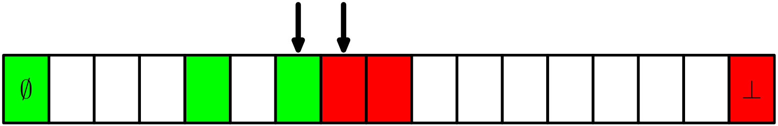

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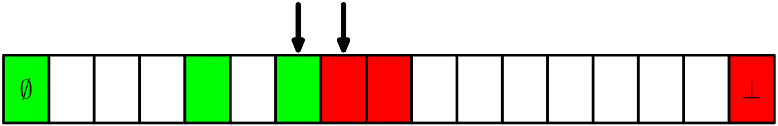

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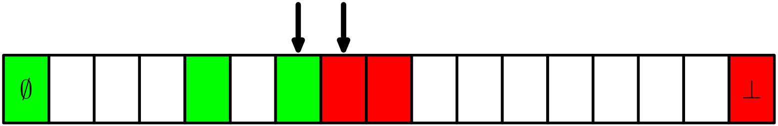

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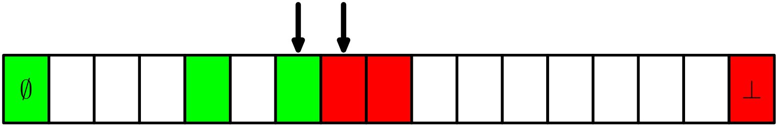

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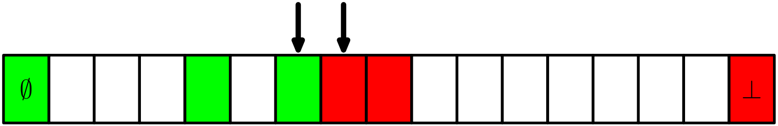

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(E.g. for polynomial calculus Alice and Bob simply evaluate their part of each monomial and exchange values — cutting planes bit more involved but can be done)

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Construct hard communication problems by "hardness amplification" using lifting

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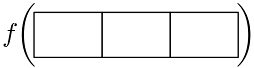

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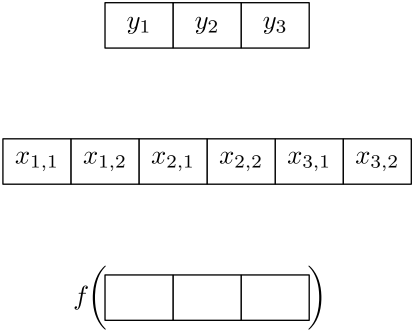

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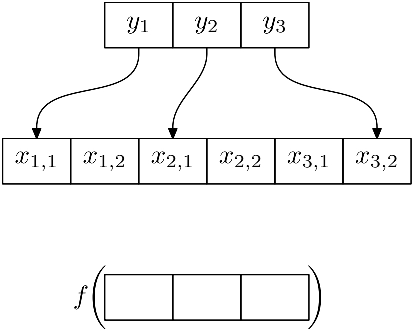

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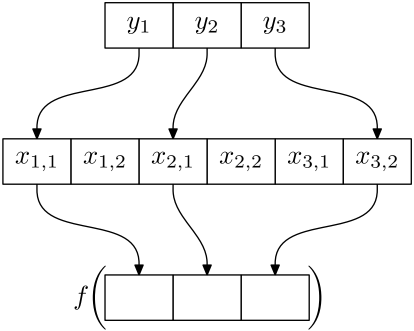

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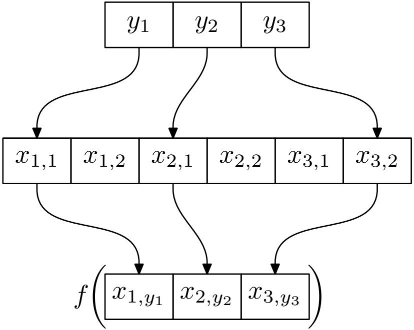
Length-ℓ lifting of f defined as
$Lift_\ell(f)(x,y) := f(x_{1,y_1}, \ldots, x_{m,y_m})$



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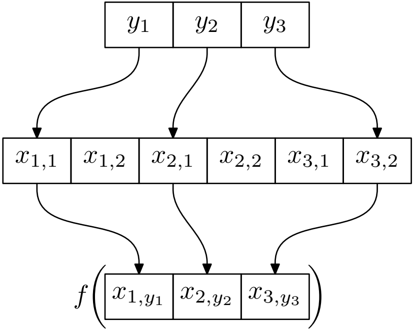

Idea borrowed from [Beame, Huynh & Pitassi '10]

Critical Block Sensitivity of Search Problems

- Block sensitivity of f on α: $\#$ disjoint blocks of α that flip f if flipped
- Problem: falsified clause search problem defines relation, not function
- Study block sensitivity of search problems
- In addition restrict to critical inputs (where relation is "function-like" in that there is only one right answer)
- Prove randomized communication complexity lower bounds in terms of critical block sensitivity of search problems
- Proof uses information-theoretic approach inspired by [Bar-Yossef, Jayram, Kumar & Sivakumar '04]

Communication Complexity Results

We prove two technical lemmas:

Lemma 1

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Lemma 2

Search problems for pebbling formulas constructed from specfic family of pyramid graphs have large critical block sensitivity

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- Encode lifting of search problem for CNF as new formula $Lift(F)$ (as in [Beame, Huynh & Pitassi '10])

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- Efficient refutation of $Lift(F)$
  $\Rightarrow$ efficient communication protocol for $Search(Lift(F))$

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- But communication complexity of lifted search problem lower-bounded by critical block sensitivity (Lemma 1)
- Plug in lower bound for pyramid pebbling formulas (Lemma 2)
  $\Rightarrow$ trade-off for lifted pebbling formulas

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Recently achieved for polynomial calculus in [Beck, N. & Tang '12] (using different techniques; in particular random restrictions)

Still open for cutting planes (random restrictions don't work)

Unconditional Space Lower Bounds?

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Can log *length factor be removed from results to yield unconditional space lower bounds?*

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Again answer known to be "yes" for resolution

But [Beck, N. & Tang '12] still has log factor for polynomial calculus

Underlying question: For how wide a family of proof systems do pebbling properties of graphs carry over to CNF size-space trade-offs?

Take-Home Message

- Modern SAT solvers enormously successful in practice — key issue is to minimize time and memory consumption
- Modelled by proof size and space in proof complexity
- We show trade-offs indicating that simultaneous optimization impossible for well-known algebraic and geometric proof systems
- Future theoretical work: Understand size and space in these proof systems better
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Thank you for your attention!