A One-Size-Fits-All Proof Logging System?

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Joint work with Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Andy Oertel, and Yong Kiam Tan

The Success of Combinatorial Solving (and the Dirty Little Secret)

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

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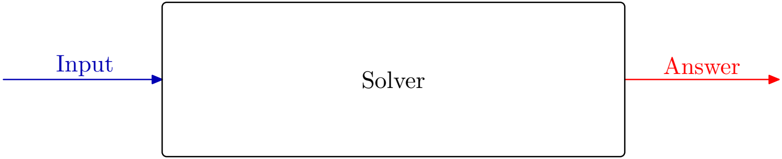
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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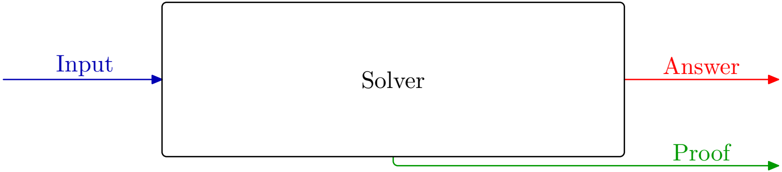

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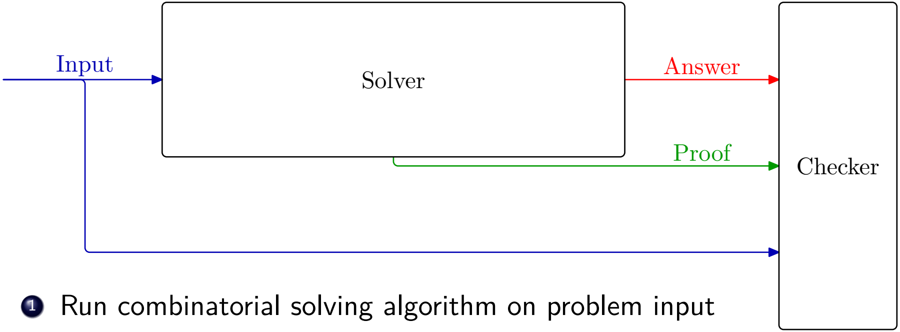

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3. Feed input + answer + proof to proof checker

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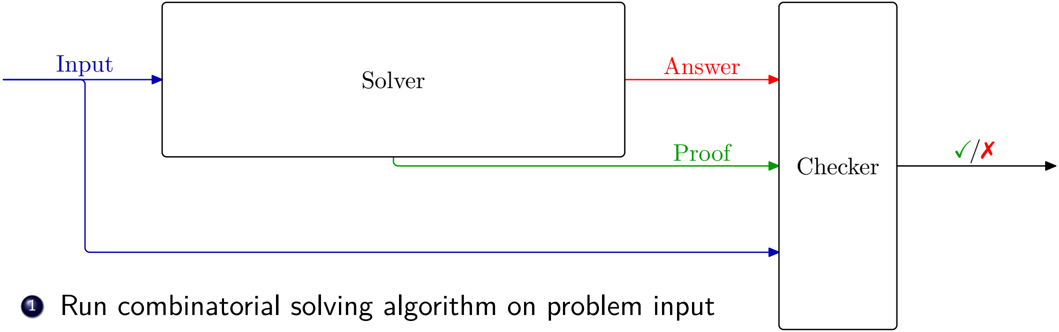
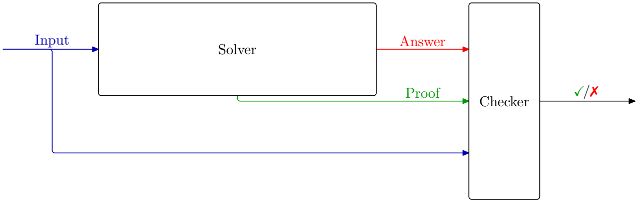


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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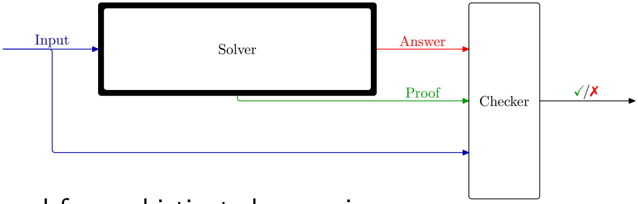

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- **very powerful:** minimal overhead for sophisticated reasoning

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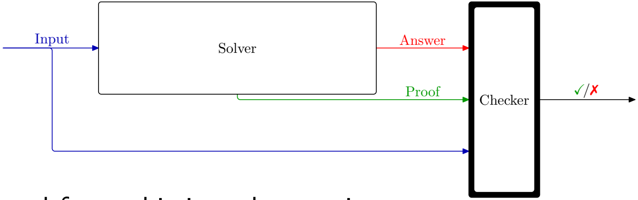

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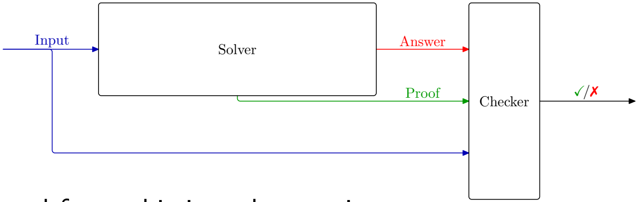

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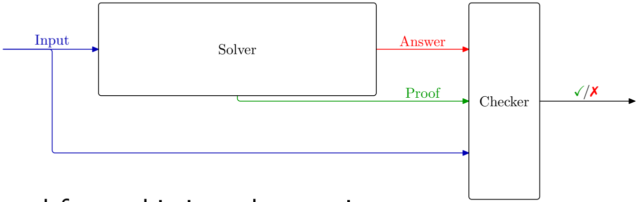

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Asking for both perhaps a little bit too good to be true?

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Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN22]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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Pseudo-Boolean Constraints

0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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If problem is (special case of) 0-1 integer linear program

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Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments (even for SAT)
3. Efficient reification of constraints

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$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\qquad$ $7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\qquad$ $9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$

VeriPB Proof Structure

1. **Preamble**
   Load input formula
   Specify settings
2. **Derivation section**
   Derivations of new constraints
   Logging of solutions
3. **Output section**
   Listing of constraints currently in database
   Input to next stage (or for debugging)
4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality
   - enumeration of solutions

VeriPB Proof Structure: Syntax

```
pseudo-Boolean proof version 2.0
f ⟨M⟩
preserve ⟨var1⟩ ⟨var2⟩ ... ⟨varN⟩
⟨derivation part⟩
output ⟨output part⟩
conclusion ⟨conclusion part⟩
end pseudo-Boolean proof
```

VeriPB Proof Configuration

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
- Also intermediate constraints used in proof logging

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Objective $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or $f = 0$ for decision problem
- Keep track of best known bound; initialize to ∞

Order $\mathcal{O}$

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set $\vec{z}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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$$\frac{}{\ell_i \geq 0}$$

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i\ell_i \geq A}{\sum_i ca_i\ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)
$$\frac{\sum_i a_i\ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Mul by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Mul by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Mul by } 2\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \overline{z} \geq 0$$

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$$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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$$\frac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$$

Div by 3
$$\frac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\tfrac{1}{3}}$$

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Such a calculation can be written in a proof line assuming handles

$$\begin{aligned} C_1 &\doteq 2x + y + w \geq 2 \\ C_2 &\doteq 2x + 4y + 2z + w \geq 5 \\ Ax(\overline{z}) &\doteq \overline{z} \geq 0 \end{aligned}$$

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using postfix notation something like

$$C_1 \;\; 2 \;\; \texttt{Mul} \;\; C_2 \;\; \texttt{Add} \;\; Ax(\overline{z}) \;\; 2 \;\; \texttt{Mul} \;\; \texttt{Add} \;\; 3 \;\; \texttt{Div}$$

More About VeriPB Proofs

Variables

- start with a letter in `A-Z` or `a-z`
- continue with characters in `A-Z`, `a-z`, `0-9`, or square and curly brackets, hyphen, underscore, and caret
- contain at least two characters

Constraints

Are referred to by positive integers (constraint IDs)

Derivation rules and requirements

Come in two flavours

1. kernel format for formally verified proof checker
2. augmented format with convenience rules such as reverse unit propagation (RUP)

Strengthening Rules

Witness ω: substitution mapping variables to truth values or literals

Redundance-based strengthening (witness ω show how to "patch assignment")

Derive constraint C from $\mathcal{C} \cup \mathcal{D}$ if exists witness ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \vdash (\mathcal{C} \cup \mathcal{D} \cup \{C\})\!\upharpoonright_\omega \cup \{f\!\upharpoonright_\omega \leq f\} \cup \mathcal{O}(\vec{z}\!\upharpoonright_\omega, \vec{z})$$

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- Witness ω should be specified in proof log
- Derivations should also be explicit, or be "obvious" to proof checker (like by RUP)

Checked and Unchecked Deletion

Important to allow deletions of constraints from database

But powerful strengthening rules create problems:

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Solution: distinguish between deletion from core set $\mathcal{C}$ and derived set $\mathcal{D}$
(For SAT solvers, support generic delete command in augmented format that translates to right type of deletion behind the scenes)

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But powerful strengthening rules create problems:

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- Satisfiable formulas can turn unsatisfiable(!)

Solution: distinguish between deletion from core set $\mathcal{C}$ and derived set $\mathcal{D}$
(For SAT solvers, support generic delete command in augmented format that translates to right type of deletion behind the scenes)

Deletion of constraint C is:

1. always OK from derived set $\mathcal{D}$
2. OK from core set $\mathcal{C}$ only if C can be rederived from $\mathcal{C} \setminus \{C\}$ with redundance rule

Checked and Unchecked Deletion

Important to allow deletions of constraints from database
But powerful strengthening rules create problems:

- Unsatisfiable formulas can turn satisfiable
- Satisfiable formulas can turn unsatisfiable(!)

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Deletion of constraint C is:

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2. OK from core set $\mathcal{C}$ only if C can be rederived from $\mathcal{C} \setminus \{C\}$ with redundance rule (otherwise unchecked deletion — special conditions apply)

Conclusions for Decision Problems

`NONE`
Status is undetermined

`SAT [ :` ⟨*assignment*⟩`]`
Propagate given assignment w.r.t. database, then check against original formula
If no assignment given, then

- solution should have been logged
- no unchecked deletion must have occurred

`UNSAT [ :` ⟨*constraint ID*⟩`]`
Only valid if no solution has been logged
Check that specified constraint is contradictory (technically: negative slack)
If no constraint given, check that database unit propagates to contradiction

Optimization Problems

Any solution α found is logged with `soli` "log solution and improve" command

- α checked against current core set $\mathcal{C}$
- Objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$ added to core set (forces search for better solutions)

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Note that

- α need not be solution for original formula
- but such solution can be reconstructed from the proof

Proof format supports not just optimality, but also non-tight upper and lower bounds

Conclusions for Optimization Problems

`NONE`
No solution or lower bound found

`BOUNDS` $\langle \mathit{LB} \rangle$ `[ :` $\langle$*constraint ID*$\rangle$ `]` $\langle \mathit{UB} \rangle$ `[ :` $\langle$*assignment*$\rangle$ `]`
$\langle LB \rangle$ and $\langle UB \rangle$ are integers or `inf`; optimality if $\langle LB \rangle = \langle UB \rangle$

Conclusions for Optimization Problems

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Lower bound
Constraint $\langle$*constraint ID*$\rangle$, if specified, should imply lower bound
Otherwise, $f \geq \langle LB \rangle$ should be "obvious" to proof checker from current database

Conclusions for Optimization Problems

`NONE`
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`BOUNDS` $\langle$*LB*$\rangle$ `[ :` $\langle$*constraint ID*$\rangle$ `]` $\langle$*UB*$\rangle$ `[ :` $\langle$*assignment*$\rangle$ `]`
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Constraint $\langle$*constraint ID*$\rangle$, if specified, should imply lower bound
Otherwise, $f \geq \langle LB \rangle$ should be "obvious" to proof checker from current database

Upper bound
Propagate given assignment w.r.t. database, then check against original formula
If no assignment given, then

- solution with value $\langle UB \rangle$ should have been logged
- no unchecked deletion must have occurred

Projected Model Enumeration and Preserved Variables

Command

`preserve` $\langle \mathit{var1} \rangle$ $\langle \mathit{var2} \rangle$ ... $\langle \mathit{varN} \rangle$

in proof preamble (after loading formula) specifies set V of preserved variables

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Preserved variables cannot appear in domain of any witness ω for strengthening rules

Any solution α found is logged with "log solution and exclude" `solx` command

- α checked against current core set $\mathcal{C}$
- Solution-excluding constraint $\bigvee_{x \in V}(x \neq \alpha(x))$ added to core set (forces search for other solutions)

Conclusions for Projected Model Enumeration Problems

`NONE`
No solution or contradiction found

`ENUMERATION PARTIAL :` $\langle N \rangle$
The number of `solx` commands in the proof log is $\langle N \rangle$
No unchecked deletion must have occurred

`ENUMERATION COMPLETE :` $\langle N \rangle$ `[ :` $\langle \mathit{constraint\ ID} \rangle$ `]`
The list of solutions found and enumerated is complete
The number of `solx` commands in the proof log is $\langle N \rangle$
Check that specified constraint is contradictory (technically: negative slack)
If no constraint given, check that database unit propagates to contradiction
No unchecked deletion must have occurred

Problem Reformulation and Output Section

`NONE`
No output

`DERIVABLE`
Any unsatisfiability / lower bound shown for output will be valid also for input

`EQUI-SATISFIABLE`
Input and output are equisatisfiable
true for decision problems with checked deletion

`EQUI-OPTIMAL`
Input and output have same optimal value
(or optimal solution was found and the output is unsatisfiable)

`EQUI-ENUMERABLE`
Input and output have the same number of projected solutions
(and no solutions have been logged)

Objective Update

Objective function update command

`obju` $\langle$*constraint ID 1*$\rangle$ $\langle$*constraint ID 2*$\rangle$ `:` $\langle f_{\mathrm{new}} \rangle$

changes objective function of (potentially reformulated) problem

Specifies two constraints in core set showing $f_{\mathrm{old}} = f_{\mathrm{new}}$

- $f_{\mathrm{old}} \leq f_{\mathrm{new}}$ is implied by $\langle$*constraint ID 1*$\rangle$
- $f_{\mathrm{old}} \geq f_{\mathrm{new}}$ is implied by $\langle$*constraint ID 2*$\rangle$

Using VeriPB for SAT Solving

1. Use dedicated tools for Gaussian elimination [GN21], symmetry breaking [BGMN22], PB-to-CNF translation [GMNO22], et cetera
2. Concatenate with CDCL solver DRAT proof rewritten in VeriPB format (`https://gitlab.com/MIAOresearch/tools-and-utilities/kissat_fork`)

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Short dictionary for DRAT-to-VeriPB translations

DRAT	VeriPB
`1`	`x1`
`-2`	`~x2`
`1 -2 3 0`	`1 x1 1 ~x2 1 x3 >= 1 ;`
`1 -2 3 0` is RUP	`rup 1 x1 1 ~x2 1 x3 >= 1 ;`
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3. But LRAT syntactically rewritten for VeriPB should be way faster to check

VeriPB Documentation

VeriPB tutorial [BMN22] (video at `https://youtu.be/s_5BIi4I22w`)

And upcoming half-day tutorial at *IJCAI '23*!

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition (available at `https://satcompetition.github.io/2023/checkers.html`)

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, BGMN22, GMN22, GMNO22, VDB22, BBN+23]

Lots of concrete example files at `https://gitlab.com/MIAOresearch/software/VeriPB`

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-trim [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

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- Lots of other challenging problems and interesting ideas

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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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