MONOTONE CIRCUITS

Circuit is monotone if no NOT-gates (only AND- and OR-gates)

For $x, y \in \{0,1\}^n$, write $x \leq y$ if for all $i \in [n]$ $x_i \leq y_i$

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is MONOTONE if $\boxed{x \leq y \Rightarrow f(x) \leq f(y)}$

$\Updownarrow$

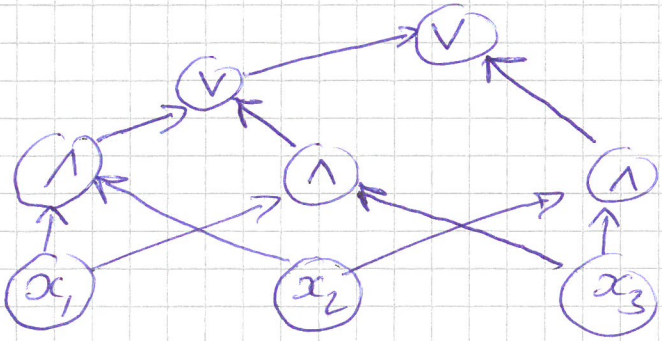
flipping any bit in input from 0 to 1 can never flip function from 1 to 0

FACTS 1

- Every monotone circuit computes a monotone function
- Every monotone function can be computed by a monotone circuit (Why?)

Ex 2 $MAJ(x_1, x_2, x_3)$ = majority value among bits

$$MAJ(x_1, x_2, x_3) = \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 \geq 2 \\ 0 & \text{o/w} \end{cases}$$



Ex 3 $\boxed{CLIQUE_{k,n}}$

Input: $\binom{n}{2}$ bits = indicator bits for edges in graph on n vertices
$\Downarrow$
specification of graph G

Output: 1 if $\exists$ k-clique in G
0 o/w

clearly monotone — adding more edges can never remove clique.

clique NP-complete $\Rightarrow$ should be hard for general circuits (unless $NP \subseteq P/poly$)

Can prove lower bound for monotone circuits:

THEOREM 4 [Razborov '85, Andreev '85, Alon-Boppana '87]
There exists a constant $\varepsilon > 0$ such that for every $k \leq n^{1/4}$ there is no monotone circuit of size less than $2^{\varepsilon\sqrt{k}}$ that computes $CLIQUE_{k,n}$

No depth restrictions or anything...

Would have $NP \neq P/poly$ if w.l.o.g. best circuit for monotone function is monotone circuit. Or at least suffer at most polynomial blow-up.
NOT TRUE! Shown by Razborov.

Consider first special kind of subfunctions / subcircuits

For $S \subseteq [n]$, let $C_S : \{0,1\}^{\binom{n}{2}} \to \{0,1\}$ be

$$C_S(G) = \begin{cases} 1 & \text{if } S \text{ forms clique in } G \\ 0 & \text{o/w} \end{cases}$$

CLIQUE INDICATOR of S

$$CLIQUE_{k,n} = \bigvee_{\substack{S \subseteq [n] \\ |S| = k}} C_S$$

$\binom{n}{k} \leq n^k$ clique indicators

Show $CLIQUE_{k,n}$ cannot be computed by OR of $< n^{\sqrt{k}/20}$ clique indicators

Idea: Create distributions of yes-instances Y and no-instances N. Show that circuit cannot tell apart random samples from the two distributions.

Distribution Y: Choose $K \subseteq [n]$, $|K| = k$ at random. Output graph with

$V = [n]$

$E = \{(u,v) \mid u \neq v,\ u,v \in K\}$

clique on K and no other edges

Distribution N: Choose function $c : [n] \to [k-1]$ at random. Output graph with all edges (u,v) for $c(u) \neq c(v)$ (i.e., complete $(k-1)$-partite graph)

$V = [n]$

$E = \{(u,v) \mid c(u) \neq c(v)\}$

$$\Pr_{G \sim \mathcal{Y}}\left[CLIQUE_{k,n}(G) = 1 \right] = 1$$

$$\Pr_{G \sim \mathcal{N}}\left[CLIQUE_{k,n}(G) = 0 \right] = 1$$

But OR of at least $n^{\sqrt{k}/20}$ clique indicators needed

LEMMA 5 Pick n sufficiently large and let $k \leq n^{1/4}$ and $S \subseteq [n]$. Then either

$$\Pr_{G \sim \mathcal{N}}\left[C_S(G) = 1 \right] \geq 0.99 \quad \frac{99}{100}$$

or

$$\Pr_{G \sim \mathcal{Y}}\left[C_S(G) = 1 \right] \leq \frac{1}{100} n^{-\sqrt{k}/20}.$$

Proof By case analysis over $|S|$.

Let $\ell = \sqrt{k-1}/10$

Case 1: $|S| \leq \ell$

By the birthday paradox bound,

$\Pr[c : [S] \to [k-1] \text{ one-to-one}] \geq 99/100$

But if c one-to-one on S we get ℓ-clique

Hence, $\Pr[C_S(\mathcal{N}) = 1] \geq 99/100$ in this case

BIRTHDAY PARADOX

MC V

Choose ℓ random numbers $a_1, \ldots, a_\ell$ uniformly and independently in $[m]$

What is the expected number of collisions? (Pairs $i<j$ such that $a_i \leq a_j$) Let

$$Y_{ij} = \begin{cases} 1 & \text{if } a_i = a_j \\ 0 & \text{o/w} \end{cases}$$

$$\mathbb{E}[Y_{ij}] = 1/m$$

$$\#\text{ collisions} = \sum_{i<j} Y_{ij}$$

$$\mathbb{E}\left[\sum_{i<j} Y_{ij}\right] = \sum_{i<j} \mathbb{E}[Y_{ij}] = \boxed{\binom{\ell}{2}\frac{1}{m}}$$

If $\ell \approx \sqrt{2m}$ expect to see collision

In class of 27 students, expect to see 2 with the same birthday

If $\ell = \sqrt{m}/K$, then $\Pr[\text{collision}] \leq \frac{1}{K^2}$

Expect to see

$$\binom{\ell}{2}\frac{1}{m} \leq \frac{\ell^2}{m} = \frac{1}{K^2} \quad \text{collision}$$

1 collision = Factor K^2 more than expected

Markov $\Pr[X \geq K] \leq \frac{\mathbb{E}[X]}{K}$

$$\Pr[\text{collision}] \leq \frac{1/K^2}{1} = 1/K^2$$

Case 2: $|S| > \ell$

For $G \sim \gamma$ $C_S(G) = 1$ iff it holds for randomly sampled $K \subseteq [n]$, $|K| = k$, that $S \subseteq K$

$$\Pr_K[S \subseteq K] \leq \frac{\binom{n-\ell}{k-\ell}}{\binom{n}{k}} =$$

$$= \frac{(n-\ell)!}{(k-\ell)!(n-\ell-k+\ell)!} \cdot \frac{k!\,(n-k)!}{n!}$$

$$= \frac{k(k-1)\cdots(k-\ell+1)}{n(n-1)\cdots(n-\ell+1)}$$

$$\leq \frac{k^\ell}{(n-\ell+1)^\ell} \leq \left[\begin{array}{c}\ell \text{ small} \\ \text{compared to } k \text{ \& } n\end{array}\right]$$

$$\leq \left(\frac{2k}{n}\right)^\ell \qquad \left[\begin{array}{l} k \leq n^{1/4} \\ \ell \geq \sqrt{k-1}/10 \end{array}\right]$$

$$\leq \left(2\, n^{-3/4}\right)^{\frac{\sqrt{k-1}}{10}}$$

$$\leq \frac{1}{100}\, n^{-\sqrt{k}/20} \qquad \text{for } n \text{ large enough}$$

Lemma 5 follows ▨

This implies that OR of too few clique indicators fails dramatically to distinguish γ and $\mathcal{N}$

COROLLARY 6

Suppose that $C' = \bigvee_{i=1}^{m} C_{S_i}$ and suppose n is large enough and $k \le n^{1/4}$.

Then if # clique indicators $m \le n^{\sqrt{k}/20}$ it holds that not only does C' fail to compute $CLIQUE_{k,n}$ but C' errs on 99% of either Y or N.

Proof If some $C_{S_{i^*}}$ has $|S_{i^*}| \le \sqrt{k-1}/10$, then

$$\Pr_{G \sim N}\left[\bigvee_{i=1}^{m} C_{S_i}(G) = 1\right] \ge \Pr\left[C_{S_{i^*}}(G) = 1\right] \ge 99/100$$

So suppose all S_i have $|S_i| > \sqrt{k-1}/10$

Then

$$\Pr_{G \sim Y}\left[\bigvee_{i=1}^{m} C_{S_i}(G) = 1\right] \le \sum_{i=1}^{m} \Pr\left[C_{S_i}(G) \neq 1\right]$$

$$\le m \cdot n^{-\sqrt{k}/20}/100$$

$$\le 1/100. \qquad \square$$

Now prove:

Monotone, small circuit C for $CLIQUE_{k,n}$

$\Downarrow$

OR of somewhat small # clique indicators distinguish Y and N well (almost as well as C)

Here formally:

LEMMA 7 Let C monotone circuit of size $s \leq 2^{\sqrt{k}/2}$. Then there exists a collection $S_1, \ldots, S_m$; $S_i \subseteq [n]$, $m \leq n^{\sqrt{k}/20}$, such that

$$\Pr_{G \sim Y}\left[\bigwedge_{i=1}^{m} C_{S_i}(G) \geq C(G)\right] > 0.9$$

$$\Pr_{G \sim N}\left[\bigwedge_{i=1}^{m} C_{S_i}(G) \leq C(G)\right] > 0.9$$

But we know $\bigwedge_{i=1}^{m} C_{S_i}$ either far too pessimistic on instances in Y or far too optimistic on instances in N.

The same must then hold for G, which hence cannot be deciding $CLIQUE_{k,n}$